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Sets, Relations & Functions

Assignment P1.1 with answer key and solutions

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ASSIGNMENT P1.1

Sets, Relations & Functions: First Look

Focus Topics: *Sets and set operations, relations and equivalence classes, domain and range, greatest integer and fractional part functions, one-one and onto maps, inverse and composite functions, functional equations, and even, odd and periodic functions.*

Section A: Single Correct Choice

[Q1–Q4] Each question has four options (A), (B), (C) and (D). Exactly ONE option is correct.

Q 1. Let $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$. The total number of injective (one-one) functions $f : S \to S$ satisfying the functional relation $f(f(x)) = 9 - x$ for all $x \in S$ is:

(A) 0 **(B)** 12 **(C)** 24 **(D)** 48

Q 2. Let $A = \{1, 2, 3, 4, 5, 6\}$. The number of distinct equivalence relations R on A such that $(1, 2) \in R$ and $(3, 4) \in R$ while $(1, 3) \notin R$ is:

(A) 31 **(B)** 34 **(C)** 10 **(D)** 45

Q 3. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be a function satisfying the identity:

$$2f(x) + 3f\left(\frac{1}{x}\right) = \frac{x^2 + 5}{x} \quad \text{for all } x \neq 0.$$

If $S = \{x \in \mathbb{R} \setminus \{0\} : f(x) = f(-x)\}$, then the sum of the squares of all elements in S is:

(A) 0 **(B)** 2 **(C)** 5 **(D)** $\frac{14}{13}$

Q 4. Let $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 2, 3\}$. The total number of surjective (onto) functions $f : A \to B$ such that $|f^{-1}(\{1\})| \geq 2$ is:

(A) 450 **(B)** 360 **(C)** 510 **(D)** 540

Section B: One or More than One Correct

[Q5–Q9] Each question has four options (A), (B), (C) and (D). ONE OR MORE options may be correct.

Q 5. Let $U = \{1, 2, 3, \ldots, 100\}$. Define $A = \{x \in U : \gcd(x, 6) = 1\}$ and $B = \{x \in U : x \text{ is a multiple of } 4\}$. Let $\mathcal{P}(X)$ denote the power set of X. Which of the following statements is/are correct?

(A) $n(A) = 33$ **(B)** $n(A \cap B) = 0$

(C) $n(A \cup B) = 58$ **(D)** $n\Big(\mathcal{P}(A \cap B)\Big) = 1$

Q 6. On the set of all integers $\mathbb{Z}$, a binary relation R is defined by:

$$a\,R\,b \iff a^3 - b^3 \text{ is divisible by } 7.$$

Which of the following statements is/are correct?

(A) R is an equivalence relation.

(B) The number of distinct equivalence classes of R is 3.

(C) If $[a]$ denotes the equivalence class containing a, then $[2] = [5]$.

(D) $[1]$ contains both 6 and 8.

Q 7. Let $f(x) = \dfrac{x}{1 + x2^{1/x}}$ for $x \in \mathbb{R} \setminus \{0\}$ satisfying $1 + x2^{1/x} \neq 0$, and let $g(x) = \lfloor x \rfloor \{x\}$ (where $\lfloor \cdot \rfloor$ and $\{\cdot\}$ denote the greatest integer and fractional part functions, respectively). Which of the following statements is/are correct?

(A) $\lim\limits_{x \to 0^+} f(x) = 0$ and $\lim\limits_{x \to 0^-} f(x) = 0$.

(B) The range of $g(x)$ for $x \in [0, \infty)$ is $[0, \infty)$.

(C) The range of $g(x)$ for $x \in (-\infty, 0]$ is $(-\infty, 0]$.

(D) The function $g(x)$ is surjective from $\mathbb{R}$ to $\mathbb{R}$.

Q 8. Let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ denote the floor and ceiling functions, respectively. Which of the following identities hold for all real numbers $x, y \in \mathbb{R}$?

(A) $\lceil x \rceil - \lfloor x \rfloor = 0$ if and only if $x \in \mathbb{Z}$.

(B) $\lfloor x \rfloor + \lfloor -x \rfloor = \lceil x \rceil + \lceil -x \rceil$ for all $x \in \mathbb{R}$.

(C) $\lfloor x + y \rfloor + \lfloor x - y \rfloor \leq 2\lfloor x \rfloor$.

(D) $\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor$.

Q 9. Let $f(x) = (-1)^{\lfloor 2x \rfloor} \left(\{2x\} - \frac{1}{2}\right)$ and $h(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$. Which of the following statements is/are correct?

(A) $f(x)$ is an even function on $\mathbb{R}$.

(B) The fundamental period of $f(x)$ is 1.

(C) $h(x)$ is periodic with fundamental period 2π.

(D) The range of $h(x)$ is $[0, \pi]$.

Section C: Numerical Answer Type

[Q10–Q15] The answer to each question is a numerical value: an integer, or a decimal rounded off to two decimal places.

Q 10. Let $A = \{1, 2, 3, 4, 5\}$. The number of functions $f : A \to A$ such that the composition satisfies $f(f(x)) = x$ for all $x \in A$ is _____.

Q 11. The number of equivalence relations on the set $\{1, 2, 3, 4, 5\}$ that contain the subset $\{(1, 2), (2, 3)\}$ is _____.

Q 12. Let $f(x) = \sqrt{\log_{1/2}\left(\dfrac{5x - x^2}{4}\right)} + \dfrac{1}{\sqrt{\{x\} - 0.5}}$, where $\{x\}$ denotes the fractional part of x. If the domain of f is $(a, b) \cup (c, d)$, then the value of $10(a + b + c + d)$ is _____.

Q 13. Let $\lfloor t \rfloor$ denote the greatest integer function. Let S be the sum of all real solutions of the equation

$$x^2 - 5\lfloor x \rfloor + 6 = 0.$$

Then $\lfloor S \rfloor$ is equal to _____.

Q 14. The number of functions $f : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ satisfying $f(f(f(x))) = f(x)$ for all $x \in \{1, 2, 3, 4, 5\}$ and having an image of size exactly 3 is _____.

Q 15. Let $f : \mathbb{R} \setminus \{0, 1\} \to \mathbb{R}$ satisfy the equation:

$$f(x) + f\left(\frac{x - 1}{x}\right) = 1 + x \quad \text{for all } x \in \mathbb{R} \setminus \{0, 1\}.$$

Then the value of $2 \cdot f(2)$ is _____.

Section D: Linked Comprehension Type

[Q16–Q17] Each passage is followed by two linked questions.

Q 16. **Passage**

> Let $X = \{1, 2, 3, 4, 5, 6, 7\}$ and $Y = \{1, 2, 3, 4\}$. A function $g : X \to Y$ is called a *balanced-type surjection* if g is onto and there exists at least one element $y \in Y$ such that $|g^{-1}(\{y\})| = 2$.

Answer the following:

16.1 The total number of surjective functions $g : X \to Y$ such that the pre-image set of the specific target element $1 \in Y$ has cardinality exactly 2 (i.e., $|g^{-1}(\{1\})| = 2$) is _____.

16.2 The total number of injective functions $h : Y \to X$ such that the product $h(1) \cdot h(2) \cdot h(3) \cdot h(4)$ is an even integer is _____.

Q 17. Passage

> Let $A = \{1, 2, 3, 4\}$. A binary relation R on A is called a *quasi-order* (or pre-order) if it is reflexive and transitive. If, in addition, R is symmetric, it is an equivalence relation; if R is antisymmetric, it is a partial order.

Answer the following:

17.1 The total number of equivalence relations on A is _____.

17.2 Let $\mathcal{E}$ be the set of all equivalence relations on A. The number of relations $R \in \mathcal{E}$ such that the partition induced by R contains at least one singleton equivalence class is _____.

Section E: Matrix Match Type

[Q18–Q19] Match each item in Column-I with all the correct entries in Column-II. An item may match one or more entries.

Q 18. Let $\lfloor\cdot\rfloor$, $\lceil\cdot\rceil$, and $\{\cdot\}$ denote the floor, ceiling, and fractional part functions, and let $\operatorname{sgn}(\cdot)$ denote the signum function. Match each function in Column-I with all properties it satisfies in Column-II.

Column I	Column II
(A) $f(x) = \ln\left(\sqrt{x^2+1} - x\right) + \operatorname{sgn}(x)\lvert x\rvert$	(p) $f(x)$ is an even function on its domain.
(B) $f(x) = \lceil x\rceil - \lfloor x\rfloor$	(q) $f(x)$ is an odd function on its domain.
(C) $f(x) = x\left(\dfrac{2^x+1}{2^x-1}\right)$	(r) $f(x)$ is periodic with fundamental period $T = 1$
(D) $f(x) = \{x\} - \{-x\}$	(s) Domain is symmetric about the origin, but $D \neq \mathbb{R}$.
	(t) The range is a subset of $(-1, 1]$

Q 19. Match the combinatorial and mapping problems in Column-I with their corresponding values in Column-II.

Column I	Column II
(A) The number of surjective functions from $\{1,2,3,4,5\}$ to $\{a,b,c\}$	(p) 60
(B) The number of idempotent functions ($f \circ f = f$) from $\{1,2,3\}$ to $\{1,2,3\}$	(q) 150
(C) The number of partitions of $\{1,2,3,4,5\}$ into exactly 2 non-empty subsets	(r) 10
(D) The number of strictly increasing functions from $\{1,2,3\}$ to $\{1,2,3,4,5\}$	(s) 15
	(t) 17

■

Answer Key: Assignment P1.1

ANSWER KEY

1. B
2. C
3. D
4. B
5. ABCD
6. AB
7. ABCD
8. AD
9. ABCD
10. 26
11. 5
12. 110
13. 5
14. 360
15. 1.5
16. **16.1** 3150
    16.2 816
17. **17.1** 15
    17.2 11
18. (A) → (q) (B) → (p, r, t)
    (C) → (p, s) (D) → (q, r, t)
19. (A) → (q) (B) → (r)
    (C) → (s) (D) → (r)

Hints and Solutions: Assignment P1.1

Q1. (B)

Let $g(x) = 9 - x$. The relation $f \circ f = g$ implies $f^4 = \mathrm{id}$ and $f \circ g = g \circ f$. The involution g consists of the four transpositions

$$(1\ 8)(2\ 7)(3\ 6)(4\ 5).$$

A permutation whose square is this product cannot contain fixed points or 2-cycles; its cycles must be 4-cycles, and each 4-cycle pairs two of the displayed transpositions. There are 3 ways to pair the four transpositions, and for each pairing there are 2 orientations for each of the two 4-cycles. Hence the number is

$$3 \cdot 2^2 = 12.$$

Therefore the correct option is (B).

Q2. (C)

An equivalence relation corresponds to a partition. The conditions $(1, 2) \in R$ and $(3, 4) \in R$ force $1, 2$ into one block and $3, 4$ into another. Since $(1, 3) \notin R$, these two blocks must remain distinct. Contract the pairs $\{1, 2\}$ and $\{3, 4\}$ to two labelled units; together with 5 and 6, we are partitioning four units while keeping the first two in different blocks. The number is

$$B_4 - B_3 = 15 - 5 = 10,$$

so the correct option is (C).

Q3. (D)

Write $A = f(x)$ and $B = f(1/x)$. The given equation and the same equation with x replaced by $1/x$ give

$$2A + 3B = x + \frac{5}{x}, \qquad 3A + 2B = 5x + \frac{1}{x}.$$

Eliminating B yields

$$f(x) = A = \frac{13x - 7/x}{5}.$$

This is odd, so $f(x) = f(-x)$ if and only if $f(x) = 0$. Thus $13x - 7/x = 0$, or $x^2 = 7/13$. The set is $S = \{\pm\sqrt{7/13}\}$, and

$$\sum_{x \in S} x^2 = \frac{7}{13} + \frac{7}{13} = \frac{14}{13}.$$

Hence the correct option is (D).

Q4. (B)

Let $k = |f^{-1}(\{1\})|$. For a surjection, k can be $1, 2, 3, 4$. We need $k \geq 2$, and after

assigning the k elements sent to 1, the remaining $6-k$ elements must map onto $\{2,3\}$. Therefore the number is

$$\sum_{k=2}^{4}\binom{6}{k}(2^{6-k}-2) = 15(14)+20(6)+15(2) = 360.$$

The correct option is (B).

Q5. (A, B, C, D)

By inclusion-exclusion,

$$n(A) = 100-50-33+16 = 33.$$

Every multiple of 4 is even, so $A \cap B = \varnothing$. Hence $n(A\cap B)=0$, $n(A\cup B) = 33+25 = 58$, and $n(\mathcal{P}(A\cap B)) = n(\mathcal{P}(\varnothing)) = 1$. Thus all four statements are correct.

Q6. (A, B)

The relation is equivalent to $a^3 \equiv b^3 \pmod 7$. The possible cubic residues modulo 7 are $0, 1, 6$, so equality of cubes modulo 7 is an equivalence relation with exactly three classes:

$$[0] = \{x : x \equiv 0 \pmod 7\}, \quad [1] = \{x : x \equiv 1,2,4 \pmod 7\},$$

$$[6] = \{x : x \equiv 3,5,6 \pmod 7\}.$$

Since $2^3 \equiv 1 \pmod 7$ and $5^3 \equiv 6 \pmod 7$, $[2] \neq [5]$. Also $8 \in [1]$ but $6 \notin [1]$. Thus only (A) and (B) are correct.

Q7. (A, B, C, D)

For $x \to 0^+$, $2^{1/x} \to \infty$, so $x2^{1/x} \to \infty$ and $f(x) \to 0$. For $x \to 0^-$, $2^{1/x} \to 0$, and again $f(x) \to 0$. (As clarified in the question statement, the formula's domain excludes any nonzero point where its denominator vanishes.)

For $g(x) = \lfloor x \rfloor \{x\}$, on $[n, n+1)$ with integer $n \geq 0$ its range is $[0, n)$; their union is $[0,\infty)$. For integer $n < 0$, the range on $[n, n+1)$ is $(n, 0]$, whose union is $(-\infty, 0]$. Therefore $g(\mathbb{R}) = \mathbb{R}$. All four statements are correct.

Q8. (A, D)

For any real x, $\lceil x \rceil - \lfloor x \rfloor = 0$ exactly when x is an integer, so (A) is true. For non-integer x, $\lfloor x \rfloor + \lfloor -x \rfloor = -1$ while $\lceil x \rceil + \lceil -x \rceil = 1$, so (B) is false. (C) is false; for example, $x = y = 1/2$ gives $\lfloor x+y \rfloor + \lfloor x-y \rfloor = 1 > 0 = 2\lfloor x \rfloor$. Finally, (D) is the standard floor identity

$$\lfloor 3x \rfloor = \lfloor x \rfloor + \left\lfloor x + \frac{1}{3} \right\rfloor + \left\lfloor x + \frac{2}{3} \right\rfloor.$$

Thus (A) and (D) are correct.

Q9. (A, B, C, D)

Put $t = 2x$. The fractional part is unchanged by adding an integer, while the parity factor changes sign after shifting x by $1/2$. Directly, using $\{-t\} = 1 - \{t\}$

for non-integer t (and checking integer t separately) shows $f(-x) = f(x)$; hence (A) is true. Shifting x by 1 increases $\lfloor 2x \rfloor$ by 2 and leaves $\{2x\}$ unchanged, so 1 is a period. The alternating linear pieces show that no smaller positive period works; (B) is true.

For $h(x) = \cos^{-1}(\cos x) - \sin^{-1}(\sin x)$, on one 2π-period it is 0 on $[0, \pi/2]$, increases from 0 to π on $[\pi/2, \pi]$, equals π on $[\pi, 3\pi/2]$, and decreases from π to 0 on $[3\pi/2, 2\pi]$. Thus h has fundamental period 2π and range $[0, \pi]$. (C) and (D) are also true.

Q10. 26

An involution on a finite set consists only of fixed points and disjoint transpositions. Let I_n be the number of involutions on an n-element set. Fixing one element either as a singleton or pairing it with one of the other $n-1$ elements gives $I_n = I_{n-1} + (n-1)I_{n-2}$. With $I_0 = I_1 = 1$, we obtain $I_2 = 2$, $I_3 = 4$, $I_4 = 10$, and $I_5 = 26$.

Q11. 5

Transitivity forces $1, 2, 3$ into the same equivalence class. Contract them into one unit; the remaining units are this block, 4, and 5. Equivalence relations correspond to partitions of these three units, so the number is the Bell number $B_3 = 5$.

Q12. 110

The first radical requires

$$\log_{1/2}\left(\frac{5x - x^2}{4}\right) \geq 0 \iff 0 < \frac{5x - x^2}{4} \leq 1,$$

which gives $x \in (0, 1] \cup [4, 5)$. The denominator of the second term requires $\{x\} - \frac{1}{2} > 0$. Intersecting these conditions gives

$$D = \left(\frac{1}{2}, 1\right) \cup \left(\frac{9}{2}, 5\right).$$

Thus $(a, b, c, d) = (1/2, 1, 9/2, 5)$, and $10(a + b + c + d) = 110$.

Q13. 5

Let $n = \lfloor x \rfloor$. Then $x^2 = 5n - 6$, so $n \geq 2$. Since $n \leq x < n + 1$, we must have

$$n^2 \leq 5n - 6 < (n + 1)^2.$$

The left inequality gives $(n - 2)(n - 3) \leq 0$, so $n \in \{2, 3\}$; both satisfy the right inequality. These yield $x = 2$ and $x = 3$, respectively. Hence $S = 2 + 3 = 5$ and $\lfloor S \rfloor = 5$.

Q14. 360

Let $Y = \mathrm{Im}(f)$ with $|Y| = 3$. If $y = f(x) \in Y$, then $f(f(y)) = f(f(f(x))) = f(x) = y$, so $f|_Y$ is an involution of Y. There are $I_3 = 4$ such involutions. Choose Y in $\binom{5}{3} = 10$ ways, choose the involution on it in 4 ways, and map each of the two elements outside Y to any element of Y in 3^2 ways. Therefore the number is

$$\binom{5}{3} \cdot 4 \cdot 3^2 = 360.$$

Q15. 1.5

Define $T(x) = (x-1)/x$. The values $2, 1/2, -1$ form a 3-cycle under T: $T(2) = 1/2$, $T(1/2) = -1$, and $T(-1) = 2$. The functional equation gives

$$f(2) + f(1/2) = 3, \quad f(1/2) + f(-1) = \frac{3}{2}, \quad f(-1) + f(2) = 0.$$

Adding the first and third equations and subtracting the second yields $2f(2) = 3 + 0 - 3/2 = 3/2 = 1.5$.

Passage 16

16.1: Choose the two elements of X that map to 1 in $\binom{7}{2} = 21$ ways. The remaining five elements must map onto the other three target elements. By inclusion-exclusion, the number of surjections from a 5-element set onto a 3-element set is $3^5 - \binom{3}{1}2^5 + \binom{3}{2}1^5 = 150$. Hence the answer is $21 \cdot 150 = 3150$.

16.2: There are $7 \cdot 6 \cdot 5 \cdot 4 = 840$ injections from Y to X. A product is odd precisely when all four images are odd; the four odd elements of X must then be used, in $4! = 24$ orders. Thus the number with even product is $840 - 24 = 816$.

Passage 17

17.1: Equivalence relations on a four-element set correspond one-to-one with its partitions. Their number is the Bell number $B_4 = 15$.

17.2: Partitions of a 4-element set with no singleton blocks have block-size types $2+2$ or 4. There are 3 partitions of type $2+2$ and 1 partition of type 4, hence 4 partitions without singletons. The number with at least one singleton block is $15 - 4 = 11$.

Q18 & Q19

Q18. (A) Since $\ln(\sqrt{x^2+1} - x) = -\operatorname{arsinh} x$, the function is $x - \operatorname{arsinh} x$, which is odd; match (A) to (q).
(B) The function $\lceil x \rceil - \lfloor x \rfloor$ is even, has period 1, and range $\{0, 1\} \subset (-1, 1]$; match (B) to (p,r,t).
(C) Its domain is $\mathbb{R} \setminus \{0\}$, symmetric about the origin, and $f(-x) = f(x)$, so it is even and matches (C) to (p,s). **(D)** The identity $\{x\} - \{-x\}$ is odd, has fundamental period 1, and range $(-1, 1)$; match (D) to (q,r,t).

Q19.
(A) Surjections from a 5-element set onto a 3-element set: $3!S(5,3) = 150$, so (A) matches (q).
(B) An idempotent map is determined by its image, all of whose elements are fixed; the count is $\sum_{k=1}^{3} \binom{3}{k} k^{3-k} = 3 + 6 + 1 = 10$, so (B) matches (r).
(C) Partitions of a 5-element set into two nonempty blocks number $S(5,2) = 15$, so (C) matches (s).
(D) A strictly increasing map from a 3-element chain into a 5-element chain is determined by choosing its three-element image, giving $\binom{5}{3} = 10$; (D) matches (r).