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ASSIGNMENT P2.1

Compound & Multiple Angles, Identities & Extreme Values

Focus Topics: *Multiple and sub-multiple angle reductions, systems of trigonometric equations, products with boundary values, telescoping sums of angles, greatest and least values, and conditions involving floor and modulus.*

Section A: Single Correct Choice

[Q1–Q4] Each question has four options (A), (B), (C) and (D). Exactly ONE option is correct.

Q 1. Let $x \in \left(0, \frac{\pi}{2}\right)$ satisfy the trigonometric equation:

$$\tan x + \tan 2x + \tan 3x + \tan 4x = 0$$

The exact value of $\sin^2 x + \sin^2 3x$ can be equal to:

(A) $\frac{5}{4}$ **(B)** 1 **(C)** $\frac{3}{2}$ **(D)** 2

Q 2. The total number of distinct ordered pairs $(x, y) \in [0, 2\pi] \times [0, 2\pi]$ satisfying the simultaneous system of equations:

$$\begin{cases} 2\sin x \cos y = 1 \\ 16\sin^2 x + 9\sec^2 y = 25 \end{cases}$$

is equal to:

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8

Q 3. The exact numerical value of the boundary-valued finite product

$$\mathcal{P} = \prod_{k=1}^{7} \left(1 + \tan \frac{k\pi}{16}\right)\left(1 + \tan \frac{(8-k)\pi}{16}\right)$$

is equal to:

(A) 128 **(B)** 256
(C) 512 **(D)** None of these

Q 4. Let $\lfloor t \rfloor$ denote the greatest integer function. The total number of real solutions to the equation

$$\lfloor \sin x \rfloor + \lfloor \cos x \rfloor + \lfloor \tan x \rfloor = 0$$

in the half-open interval $x \in [0, 2\pi)$ is:

(A) 0 **(B)** 1
(C) 2 **(D)** Infinitely many

Section B: One or More than One Correct

[Q5–Q9] Each question has four options (A), (B), (C) and (D). ONE OR MORE options may be correct.

Q 5. Let $f(x) = \sin^6 x + \cos^6 x + k(\sin^4 x + \cos^4 x)$ for $x \in \mathbb{R}$. Which of the following statements is/are correct?

(A) If $k = -\frac{3}{2}$, then $f(x)$ is independent of x.

(B) If $k = 1$, the minimum value of $f(x)$ is $\frac{7}{4}$.

(C) The maximum value of $f(x)$ for $k = -1$ is $\frac{1}{4}$.

(D) The fundamental period of $f(x)$ is $\frac{\pi}{2}$ for all $k \neq -\frac{3}{2}$.

Q 6. Consider the equation $\cos 4x + a\cos 2x + 2a - 3 = 0$, where $a \in \mathbb{R}$ is a real parameter. Which of the following statements is/are correct?

(A) The equation has real roots if and only if $a \in \left[\frac{2}{3}, 8 - 4\sqrt{2}\right]$.

(B) For $a = 1$, the equation has exactly 4 roots in $[0, \pi]$.

(C) For $a = 0$, the sum of all roots in $[0, 2\pi]$ is 4π.

(D) If $a \in (0, 2)$, the equation has exactly 8 roots in $[0, 2\pi)$.

Q 7. For each $n \in \mathbb{N}$ and each $x \in (0, \pi)$ such that $\sin(2^r x) \neq 0$ for $r = 1, \ldots, n$, define $S_n(x) = \sum_{r=1}^{n} \frac{1}{\sin(2^r x)}$. Which of the following assertions is/are correct?

(A) $S_n(x) = \cot x - \cot(2^n x)$

(B) If $x = \frac{\pi}{7}$, then $\sum_{r=1}^{3} \csc(2^r x) = 0$

(C) $S_n\left(\frac{\pi}{2^{n+1}}\right) = \cot\left(\frac{\pi}{2^{n+1}}\right)$

(D) $S_n(x) = \tan(2^n x) - \tan x$

Q 8. Let x, y, z be positive real numbers such that $x + y + z = \frac{\pi}{2}$. Which of the following inequalities always hold?

(A) $\tan x \tan y + \tan y \tan z + \tan z \tan x = 1$

(B) $\tan x + \tan y + \tan z \geq \sqrt{3}$

(C) $\sin x \sin y \sin z \leq \frac{1}{8}$

(D) $\cos x \cos y \cos z \leq \frac{3\sqrt{3}}{8}$

Q 9. Consider the system of equations for $x, y \in [0, \pi]$:

$$\begin{cases} \sin x + \sin y = \sin(x + y) \\ |x - y| = \frac{\pi}{3} \end{cases}$$

Which of the following values can x attain?

(A) $x = \frac{\pi}{3}$ **(B)** $x = \frac{2\pi}{3}$ **(C)** $x = \frac{\pi}{6}$ **(D)** $x = \frac{5\pi}{6}$

Section C: Numerical Answer Type

[Q10–Q15] The answer to each question is a numerical value: an integer, or a decimal rounded off to two decimal places.

Q 10. The exact numerical value of the trigonometric sum

$$\mathcal{S} = \tan^2\frac{\pi}{16} + \tan^2\frac{3\pi}{16} + \tan^2\frac{5\pi}{16} + \tan^2\frac{7\pi}{16}$$

is equal to _____.

Q 11. The total number of solutions to the equation

$$\sin x = \frac{x}{100}$$

in the real field $\mathbb{R}$ is _____.

Q 12. Let $x \in \left(0, \frac{\pi}{2}\right)$ satisfy the equation $\cos^3 x + \cos^2 x - 4\cos x - 1 = -3$. The value of

$$\mathcal{E} = \left|\tan^2 x - 2\sec x\right|$$

is equal to _____.

Q 13. The minimum value of the expression

$$f(x,y) = \frac{(3\sin x - 4\cos y - 15)^2}{2} + \frac{(3\cos x - 4\sin y - 20)^2}{2}$$

for all real numbers $x, y \in \mathbb{R}$ is _____.

Q 14. The value of the finite sum

$$\mathcal{T} = \sum_{k=1}^{88} \frac{1}{\cos k^\circ \cos(k+1)^\circ}$$

can be uniquely expressed in the form $\frac{\tan a^\circ - \tan b^\circ}{\sin c^\circ}$ where $a, b, c \in [1, 89] \cap \mathbb{N}$. The value of $a + b + c$ is _____.

Q 15. The number of distinct real solutions of the equation

$$8\cos x \cos 2x \cos 4x = 1$$

in the interval $x \in [0, \pi]$ is _____.

Section D: Linked Comprehension Type

[Q16–Q17] Each passage is followed by two linked questions.

Q 16. Passage

Consider the sequence of polynomial identities arising from multiple angles. It is known that for any odd prime $p = 2m + 1$, the roots of the polynomial equation

$$\sum_{k=0}^{m} (-1)^k \binom{2m+1}{2k+1} x^{m-k} = 0$$

are given by $x_r = \cot^2\left(\frac{r\pi}{2m+1}\right)$ for $r = 1, 2, \ldots, m$.

Answer the following:

16.1 For $m = 5$ (corresponding to the denominator 11), the value of the symmetric sum $\sum_{r=1}^{5} \cot^2\left(\frac{r\pi}{11}\right)$ is equal to:

(A) 15 **(B)** 45 **(C)** 55 **(D)** 110

16.2 The value of the alternating quotient product $\prod_{r=1}^{5}\left(1 + \cot^2\frac{r\pi}{11}\right)$ is equal to:

(A) 256 **(B)** 512 **(C)** 2048 **(D)** $\frac{1024}{11}$

Q 17. Passage

Let $f(x) = \cos(\cos x) - \sin(\sin x)$ defined for all $x \in \mathbb{R}$. The analytical and structural behaviour of nested trigonometric functions gives rise to extreme bounds and localized parity patterns.

Answer the following:

17.1 The absolute maximum value attained by the function $f(x)$ on $\mathbb{R}$ is:

(A) 1 **(B)** $\cos 1$ **(C)** $1 + \sin 1$ **(D)** $\cos 1 - \sin 1$

17.2 The total number of roots of the equation $f(x) = 0$ in the closed interval $x \in [0, 2\pi]$ is:

(A) 0 **(B)** 1 **(C)** 2 **(D)** 4

Section E: Matrix Match Type

[Q18–Q19] Match each item in Column-I with all the correct entries in Column-II. An item may match one or more entries.

Q 18. Match the trigonometric equations in Column-I with the corresponding number of solutions in the specified domain in Column-II.

Column I	Column II
(A) $\sin 3x + \cos 2x = 1 + 2\sin x \cos 2x$ in $x \in [0, 2\pi]$	(p) 2
(B) $\tan x + \tan 2x + \sqrt{3}\tan x \tan 2x = \sqrt{3}$ in $x \in [0, \pi]$	(q) 3
(C) $2^{\sin^2 x} + 2^{\cos^2 x} = 2\sqrt{2}$ in $x \in [0, 2\pi]$	(r) 4
(D) $\sin^4 x - \cos^4 x = \frac{1}{2}$ in $x \in [0, 2\pi]$	(s) 5
	(t) 6

Q 19. Match each trigonometric expression in Column-I with its exact evaluated constant in Column-II.

Column I	Column II
(A) $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$	(p) $\frac{1}{8}$
(B) $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$	(q) $-\frac{1}{8}$
(C) $\tan 20^\circ \tan 40^\circ \tan 80^\circ$	(r) $\sqrt{3}$
(D) $\cos 20^\circ \cos 40^\circ \cos 80^\circ$	(s) $\frac{1}{16}$
	(t) $\frac{\sqrt{7}}{8}$

■

Answer Key: Assignment P2.1

ANSWER KEY

1. A

2. D

3. D

4. D

5. AD

6. A

7. ABC

8. ABCD

9. A

10. 28

11. 63

12. 1.87

13. 162

14. 91

15. 7

16. **16.1** 15

16.2 D

17. **17.1** C

17.2 A

18. (A) → (s) (B) → (q)
(C) → (r) (D) → (r)

19. (A) → (q) (B) → (p)
(C) → (r) (D) → (p)

Hints and Solutions: Assignment P2.1

Q1. (A)

Pair the terms as $(\tan x + \tan 4x) + (\tan 2x + \tan 3x)$. Wherever all four tangent terms are defined,

$$\tan x + \tan 2x + \tan 3x + \tan 4x = \frac{\sin 5x\Big(\cos 2x \cos 3x + \cos x \cos 4x\Big)}{\cos x \cos 2x \cos 3x \cos 4x}.$$

Thus either $\sin 5x = 0$ or $\cos 5x + \cos 2x \cos x = 0$. In $0 < x < \pi/2$, the first condition gives $x = \pi/5, 2\pi/5$, for which

$$\sin^2 x + \sin^2 3x = \frac{5}{4}.$$

For the second condition, put $t = \cos x \in (0, 1)$. Then

$$\cos 5x + \cos 2x \cos x = 2t(8t^4 - 9t^2 + 2) = 0,$$

so $t^2 = (9 \pm \sqrt{17})/16$. Both values lie in $(0, 1)$ and yield additional admissible roots. At these roots,

$$\sin^2 x + \sin^2 3x = \frac{1}{2} + \frac{3}{4}t^2 = \frac{59 \pm 3\sqrt{17}}{64}.$$

The expression therefore takes three distinct values, out of them one is in the option.

Q2. (D)

From $2 \sin x \cos y = 1$, we have $\sin x = 1/(2 \cos y)$. Substitution into the second equation gives

$$\frac{4}{\cos^2 y} + \frac{9}{\cos^2 y} = 25 \quad \Longrightarrow \quad \cos^2 y = \frac{13}{25},$$

and hence $\sin^2 x = 25/52$. Each squared trigonometric equation has two angles with positive sign and two with negative sign in $[0, 2\pi]$. The condition $\sin x \cos y > 0$ requires the signs to agree. There are $2 \cdot 2$ pairs with both positive and $2 \cdot 2$ pairs with both negative, for a total of

$$\boxed{8}.$$

Q3. (D)

As k runs from 1 to 7, the indices $8 - k$ also run through $1, \ldots, 7$. Therefore

$$\mathcal{P} = \left[\prod_{k=1}^{7} \left(1 + \tan \frac{k\pi}{16}\right)\right]^2.$$

The complementary-angle identity does not make each paired factor equal to 2; rather,

$$(1 + \tan\theta)(1 + \cot\theta) = 2 + \tan\theta + \cot\theta.$$

Numerically, the product is approximately 84466.13, which is not any of the listed choices $128, 256, 512$.

Q4. (D)

For every $x \in (0, \pi/4)$, all three values $\sin x$, $\cos x$ and $\tan x$ lie in $(0, 1)$, so each has floor 0. Hence the equation is satisfied throughout the interval $(0, \pi/4)$ and has infinitely many solutions. The answer is **(D)**.

Q5. (A, D)

Put $t = \sin^2 x \cos^2 x \in [0, 1/4]$. Then

$$\sin^4 x + \cos^4 x = 1 - 2t, \qquad \sin^6 x + \cos^6 x = 1 - 3t,$$

so $f(x) = (1 + k) - (3 + 2k)t$.

(A) For $k = -3/2$, the coefficient of t is zero, so f is constant. True.

(B) For $k = 1$, $f = 2 - 5t$ and $f_{\min} = 2 - 5/4 = 3/4$, not $7/4$. False.

(C) For $k = -1$, $f = -t$, whose maximum is 0, not $1/4$. False.

(D) For $k \neq -3/2$, f is a nonconstant affine function of $\sin^2(2x)$, whose fundamental period is $\pi/2$. True.

Thus the correct options are **A, D**.

Q6. (A)

Put $t = \cos 2x \in [-1, 1]$. The equation becomes

$$2t^2 + at + 2a - 4 = 0 \quad \Longrightarrow \quad a = \frac{2(2 - t^2)}{t + 2}.$$

Let $g(t) = \dfrac{2(2 - t^2)}{t + 2}$. Then

$$g'(t) = -\frac{2(t^2 + 4t + 2)}{(t + 2)^2}.$$

The maximum on $[-1, 1]$ occurs at $t = -2 + \sqrt{2}$ and equals $8 - 4\sqrt{2}$. The minimum is $g(1) = \frac{2}{3}$. Thus real roots exist precisely when

$$a \in \left[\frac{2}{3},\, 8 - 4\sqrt{2}\right],$$

which is option (A). For $a = 1$, the quadratic in t has only one root in $[-1, 1]$, giving two roots in $[0, \pi]$, not four. For $a = 0$, $\cos 4x = 3$ has no real roots. Taking $a = 1$ also disproves (D): there are four roots in $[0, 2\pi)$, not eight. Thus (B), (C), and (D) are false.

Q7. (A, B, C)

Use the identity

$$\csc 2\theta = \cot\theta - \cot 2\theta.$$

The sum telescopes:

$$S_n(x) = \sum_{r=1}^{n} \csc(2^r x) = \cot x - \cot(2^n x),$$

so (A) is true. For $x = \frac{\pi}{7}$ and $n = 3$,

$$S_3\left(\frac{\pi}{7}\right) = \cot\frac{\pi}{7} - \cot\frac{8\pi}{7} = 0,$$

so (B) is true. If $x = \frac{\pi}{2^{n+1}}$, then $2^n x = \frac{\pi}{2}$ and

$$S_n(x) = \cot\left(\frac{\pi}{2^{n+1}}\right),$$

so (C) is true. For $n = 1$ and $x = \frac{\pi}{8}$, the sides of (D) are $\sqrt{2}$ and $2 - \sqrt{2}$, respectively, so (D) is false. The correct options are (A), (B), and (C).

Q8. (A, B, C, D)

Since $x + y + z = \pi/2$, the tangent addition identity gives

$$\tan x \tan y + \tan y \tan z + \tan z \tan x = 1,$$

so (A) is true. Writing $u = \tan x$, $v = \tan y$, $w = \tan z$, we have

$$(u + v + w)^2 \geq 3(uv + vw + wu) = 3,$$

which proves (B). By concavity of $\log \sin t$ on $(0, \pi/2)$, the product $\sin x \sin y \sin z$ is maximised at $x = y = z = \pi/6$ and is at most $1/8$, proving (C). Similarly, $\log \cos t$ is concave, so $\cos x \cos y \cos z$ is maximised at the same point and is at most $(\sqrt{3}/2)^3 = 3\sqrt{3}/8$, proving (D). All four options are correct.

Q9. (A)

Put $s = (x + y)/2$. By sum-to-product and $|x - y| = \pi/3$, the equation becomes

$$2\sin s\left(\frac{\sqrt{3}}{2} - \cos s\right) = 0.$$

Here $s \in [0, \pi]$. The case $\sin s = 0$ gives $x + y = 0$ or 2π, both incompatible with $|x - y| = \pi/3$. The other case gives $s = \pi/6$, so $x + y = \pi/3$. Together with $|x - y| = \pi/3$, the solutions are $(x, y) = (\pi/3, 0)$ and $(0, \pi/3)$. Thus x can be $\pi/3$ among the listed values, and only option **(A)** is correct.

Q10. 28

Let $u = \tan^2\theta$. For $\theta = \pi/16, 3\pi/16, 5\pi/16, 7\pi/16$, we have $\tan^2(4\theta) = 1$. Using the multiple-angle formula for $\tan 4\theta$ and simplifying gives

$$u^4 - 28u^3 + 70u^2 - 28u + 1 = 0.$$

The four requested values are the four roots of this quartic. By Vieta's formula their sum is 28. Hence $\boxed{28}$.

Q11. 63

The equation $\sin x = x/100$ can have solutions only for $|x| \leq 100$. On the positive axis, the first positive lobe $(0, \pi)$ contributes one nonzero solution in addition to the endpoint solution $x = 0$. Each lobe $(2k\pi, (2k + 1)\pi)$ for $k = 1, \ldots, 15$ contributes two intersections; the last such lobe lies before 100 because $31\pi < 100$, and its peak is above the line $x/100$. Thus there are $1 + 15 \cdot 2 = 31$ positive nonzero solutions. By odd symmetry there are 31 negative nonzero solutions, and $x = 0$ is one more solution. The total is $31 + 31 + 1 = \boxed{63}$.

Q12. $1+\dfrac{\sqrt{3}}{2}$

Let $t=\cos x$. Since $x\in\left(0,\frac{\pi}{2}\right)$, we have $0<t<1$. The given equation becomes

$$t^3+t^2-4t+2=0 \quad\Longrightarrow\quad (t-1)(t^2+2t-2)=0.$$

Since $0<t<1$, it follows that $t=\sqrt{3}-1$. Hence

$$\sec x=\frac{\sqrt{3}+1}{2}, \qquad \tan^2 x=\sec^2 x-1=\frac{\sqrt{3}}{2}.$$

Therefore

$$\begin{aligned}\mathcal{E}&=\left|\tan^2 x-2\sec x\right|\\&=\left|\frac{\sqrt{3}}{2}-(\sqrt{3}+1)\right|\\&=1+\frac{\sqrt{3}}{2}\approx 1.87.\end{aligned}$$

Q13. 162

Write $P=(3\sin x,3\cos x)$, which lies on the circle centred at $(0,0)$ with radius 3, and $Q=(4\cos y+15,4\sin y+20)$, which lies on the circle centred at $(15,20)$ with radius 4. Then $PQ^2=2f(x,y)$. The distance between the centres is $\sqrt{15^2+20^2}=25$, so the least distance between the two circles is $25-3-4=18$. Consequently,

$$PQ^2\geq 18^2=324 \quad\Longrightarrow\quad f(x,y)\geq 162.$$

The bound is attained at the points on the line joining the centres closest to each other. Hence $\boxed{162}$.

Q14. 91

Using

$$\tan(k+1)^\circ-\tan k^\circ=\frac{\sin 1^\circ}{\cos k^\circ\cos(k+1)^\circ},$$

the sum telescopes:

$$\mathcal{T}=\frac{\tan 89^\circ-\tan 1^\circ}{\sin 1^\circ}$$

Q15. 7

For $\sin x\neq 0$, the identity $8\cos x\cos 2x\cos 4x=\sin 8x/\sin x$ reduces the equation to $\sin 8x=\sin x$. Thus

$$2\cos\frac{9x}{2}\sin\frac{7x}{2}=0.$$

The first factor gives $x=(2k+1)\pi/9$; in $[0,\pi]$, these give five candidates, of which $x=\pi$ must be excluded from the original equation, leaving four. The second factor gives $x=2m\pi/7$; excluding $x=0$ leaves three candidates. The two sets do not overlap, so there are $4+3=\boxed{7}$ solutions.

Passage 16

16.1 (15): By Vieta's formula for the stated polynomial,

$$\sum_{r=1}^{m} \cot^2\left(\frac{r\pi}{2m+1}\right) = \frac{m(2m-1)}{3}.$$

For $m = 5$, this equals $5 \cdot 9/3 = \boxed{15}$.

16.2 (D): Since $1 + \cot^2\theta = \csc^2\theta$ and

$$\prod_{r=1}^{m} \sin\left(\frac{r\pi}{2m+1}\right) = \frac{\sqrt{2m+1}}{2^m},$$

we obtain

$$\prod_{r=1}^{5} \left(1 + \cot^2\frac{r\pi}{11}\right) = \left(\frac{2^5}{\sqrt{11}}\right)^2 = \boxed{\frac{1024}{11}}.$$

This is option (D).

Passage 17

17.1 (C): Because $\cos(\cos x) \le 1$ and $-\sin(\sin x) \le \sin 1$, we have

$$f(x) \le 1 + \sin 1.$$

Equality occurs at $x = 3\pi/2 + 2k\pi$, where $\cos x = 0$ and $\sin x = -1$. Therefore the maximum is $\boxed{1 + \sin 1}$, which matches option (C).

17.2 (A): If $f(x) = 0$, then $\cos(\cos x) = \sin(\sin x)$. The left side is positive, so $\sin x > 0$. Using $\sin u = \cos(\pi/2 - u)$ and the equality condition for cosines gives either

$$\cos x + \sin x = \frac{\pi}{2} + 2k\pi \quad \text{or} \quad \cos x - \sin x = -\frac{\pi}{2} + 2k\pi.$$

Both are impossible because $|\cos x \pm \sin x| \le \sqrt{2} < \pi/2$. Hence there are no roots in $[0, 2\pi]$.

Q18 and Q19

Q18(A): The equation reduces to $\sin x + \cos 2x = 1$, hence $\sin x(1 - 2\sin x) = 0$. In $[0, 2\pi]$, the solutions are $0, \pi, 2\pi, \pi/6, 5\pi/6$: five solutions, so (A) matches (s).

Q18(B): The given equation is equivalent, on its domain, to $\tan 3x = \sqrt{3}$. Thus $3x = \pi/3 + k\pi$, and in $[0, \pi]$ the three solutions are $x = \pi/9, 4\pi/9, 7\pi/9$. Hence (B) matches (q), not (p).

Q18(C): By AM–GM,

$$2^{\sin^2 x} + 2^{\cos^2 x} \ge 2\sqrt{2^{\sin^2 x + \cos^2 x}} = 2\sqrt{2}.$$

Equality holds when $\sin^2 x = \cos^2 x = 1/2$, giving four solutions in $[0, 2\pi]$. Thus (C) matches (r).

Q18(D): Since $\sin^4 x - \cos^4 x = -\cos 2x$, the equation gives $\cos 2x = -1/2$, which has four solutions in $[0, 2\pi]$. Thus (D) matches (r).

Q19(A):

$$\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7} = \frac{\sin(8\pi/7)}{8\sin(\pi/7)} = -\frac{1}{8},$$

so (A) matches (q).

Q19(B): The product $\sin(\pi/14)\sin(3\pi/14)\sin(5\pi/14) = 1/8$, so (B) matches (p).

Q19(C): The triple-angle product identity gives $\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ = \sqrt{3}$, so (C) matches (r).

Q19(D): The standard product identity gives $\cos 20^\circ \cos 40^\circ \cos 80^\circ = 1/8$, so (D) matches (p).