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Assignment P3.1 with answer key and solutions

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ASSIGNMENT P3.1

Complex Numbers, Roots of Unity & Polynomials

Focus Topics: *Roots of unity and their geometry, factorising with roots of unity, De Moivre's theorem, symmetric functions of roots (Newton's sums), location of roots, and extreme values of the modulus.*

Section A: Single Correct Choice

[Q1–Q4] Each question has four options (A), (B), (C) and (D). Exactly ONE option is correct.

Q 1. Let $z_1, z_2, \ldots, z_n$ be non-zero complex numbers of equal modulus. Which of the following statements is equivalent to

$$\operatorname{Re}\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{z_j}{z_k}\right) = 0?$$

(A) $\displaystyle\sum_{k=1}^{n} z_k = 0$ **(B)** $\displaystyle\sum_{k=1}^{n} z_k = 1$

(C) $\displaystyle\sum_{k=1}^{n} z_k \in i\mathbb{R}\setminus\{0\}$ **(D)** $|z_1 + z_2 + \cdots + z_n| = n$

Q 2. Let $z \in \mathbb{C}$ satisfy the non-linear modulus constraint $|z^2 - 2z + 4| = 2|z - 1|$. The difference between the absolute maximum and absolute minimum values of the distance $|z - 1|$ is equal to:

(A) 2 **(B)** $\sqrt{5}$ **(C)** $2\sqrt{3}$ **(D)** $\sqrt{13} - 1$

Q 3. Let $P(x) = x^4 - 4x^3 + 6x^2 - 4x = 0$ have roots z_1, z_2, z_3, z_4 in $\mathbb{C}$. The perimeter of the convex polygon whose vertices are these roots in the Argand plane is:

(A) $4\sqrt{2}$ **(B)** 8 **(C)** $4(\sqrt{2}+1)$ **(D)** $4\sqrt{3}$

Q 4. Let S denote the set of all complex numbers z satisfying the simultaneous inequalities

$$|z - 2 - 2i| \le \sqrt{2} \quad \text{and} \quad \operatorname{Re}\left(\frac{z-2}{z-2i}\right) \ge 0$$

The area of the region occupied by the set S in the complex plane is:

(A) $\dfrac{\pi}{4}$ **(B)** $\dfrac{\pi}{2}$ **(C)** $\dfrac{3\pi}{4}$ **(D)** $\dfrac{2\pi}{3} + \sqrt{3}$

Section B: One or More than One Correct

[Q5–Q9] Each question has four options (A), (B), (C) and (D). ONE OR MORE options may be correct.

Q 5. Let z_1, z_2, z_3 be three distinct complex numbers satisfying $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = z_1 z_2 z_3$. Which of the following statements is/are correct?

(A) $\text{Re}(z_1 z_2 + z_2 z_3 + z_3 z_1) = 1$

(B) $\max\{|z_1 - z_2|, |z_2 - z_3|, |z_3 - z_1|\} \geq \sqrt{3}$

(C) At least one of z_1, z_2, z_3 must equal 1.

(D) The triangle formed by z_1, z_2, z_3 is acute-angled.

Q 6. Let $P(z) = z^6+z^5+z^4+z^3+z^2+z+1 = 0$ with roots $\alpha_k = e^{i\frac{2k\pi}{7}}$ for $k = 1, 2, \ldots, 6$. Let $S_m = \sum_{k=1}^{6} \alpha_k^m$ for $m \in \mathbb{N}$. Which of the following statements is/are correct?

(A) $S_7 = 6$ **(B)** $S_{14} = 6$

(C) $\sum_{k=1}^{3} \left(\alpha_k + \frac{1}{\alpha_k}\right) = -1$ **(D)** $\prod_{k=1}^{3} \left(\alpha_k + \frac{1}{\alpha_k}\right) = -1$

Q 7. Consider the polynomial $f(z) = z^4 + az^3 + bz^2 + cz + d$ where $a, b, c, d \in \mathbb{R}$. Suppose all four roots of $f(z)$ lie on the imaginary axis $\text{Re}(z) = 0$ and are non-zero and distinct. Which of the following assertions is/are correct?

(A) $a = 0$ and $c = 0$

(B) $b > 0$ and $d > 0$

(C) $b^2 - 4d > 0$

(D) $f(z) = 0$ has no purely real roots.

Q 8. Let $z \in \mathbb{C} \setminus \{0, 1\}$ and define $w = \dfrac{z-1}{z+1}$. Which of the following statements is/are correct?

(A) If $|z| = 1$ and $z \neq -1$, then w is purely imaginary.

(B) If $\text{Re}(z) > 0$, then $|w| < 1$.

(C) If $w + \dfrac{1}{w} = 2\cos\theta$ with $\theta \in \mathbb{R}$, then $|z| = 1$ or $z \in \mathbb{R}$.

(D) The transformation $w(z)$ maps circles centred at -1 to circles centred at the origin.

Q 9. Let $z_1, z_2, z_3 \in \mathbb{C}$ represent the vertices of a triangle $\triangle ABC$ in counter-clockwise orientation. Suppose $z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1 = 0$. Which of the following statements is/are correct?

(A) $\triangle ABC$ is an equilateral triangle.

(B) $\dfrac{z_2 - z_1}{z_3 - z_1} = e^{i\frac{\pi}{3}}$ or $e^{-i\frac{\pi}{3}}$

(C) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$

(D) $z_1 + \omega z_2 + \omega^2 z_3 = 0$ where $\omega = e^{i\frac{2\pi}{3}}$

Section C: Numerical Answer Type

[Q10–Q15] The answer to each question is a numerical value: an integer, or a decimal rounded off to two decimal places.

Q 10. Let $\omega = e^{i\frac{2\pi}{7}}$. The exact value of the product

$$\mathcal{P} = \prod_{k=1}^{3} \left(2\cos\left(\frac{2k\pi}{7}\right) + 1\right)$$

is equal to _____.

Q 11. Let $z \in \mathbb{C}$ satisfy $|z - 3 - 4i| \leq 2$. The maximum value of the expression $|z+1+i|^2$ is equal to $a + b\sqrt{c}$, where $a, b, c \in \mathbb{N}$ and c is square-free. The value of $a + b + c$ is _____.

Q 12. Let $P(z) = z^8 - 1 = 0$. If S denotes the sum of the fourth powers of the distances from the point $z_0 = 2$ to all the roots of $P(z) = 0$ in the complex plane, then the value of $\dfrac{S}{10}$ is equal to _____.

Q 13. The total number of distinct complex solutions z satisfying the system

$$\begin{cases} z^{12} = 1 \\ |z - 1| \geq 1 \end{cases}$$

is equal to _____.

Q 14. Let z_1, z_2, z_3 be the roots of the cubic equation $z^3 - 6z^2 + 12z - 16 = 0$. The square of the area of the triangle formed by z_1, z_2, z_3 in the Argand plane is equal to _____.

Q 15. Let $z \in \mathbb{C}$ satisfy $|z| = 1$. The maximum possible value of the multi-term expression

$$E(z) = |z^3 - z + 2|$$

is equal to _____.

Section D: Linked Comprehension Type

[Q16–Q17] Each passage is followed by two linked questions.

Q 16. Passage

Let $n \in \mathbb{N}$ with $n \geq 3$, and let $\omega_k = e^{i\frac{2k\pi}{n}}$ for $k = 0, 1, \ldots, n-1$ denote the complete set of n-th roots of unity. The canonical polynomial decomposition

$$z^n - 1 = \prod_{k=0}^{n-1} (z - \omega_k)$$

allows evaluation of rational trigonometric sums, cyclotomic products, and multi-point potential invariants along circles concentric with the origin.

Answer the following:

16.1 The exact value of the finite rational sum $\sum_{k=1}^{n-1} \dfrac{1}{1-\omega_k}$ is equal to:

(A) $\dfrac{n-1}{2}$ **(B)** $\dfrac{n}{2}$ **(C)** $n-1$ **(D)** $\dfrac{n+1}{2}$

16.2 For $n = 9$, the value of the finite product $\prod_{k=1}^{8} (1 - \omega_k + \omega_k^2)$ is:

(A) 4 **(B)** 27 **(C)** 73 **(D)** 81

Q 17. Passage

Consider the family of quadratic forms in complex variables generated by distance constraints. Let $A(z_1)$ and $B(z_2)$ be fixed points in the Argand plane. The Apollonius locus defined by

$$\left|\frac{z - z_1}{z - z_2}\right| = \lambda \quad (\lambda > 0,\ \lambda \neq 1)$$

is a circle whose centre and radius are determined by harmonic division points. Let $z_1 = 3$ and $z_2 = -1$ with ratio $\lambda = 3$.

Answer the following:

17.1 The centre z_0 and radius R of this circle are:

(A) $z_0 = -\dfrac{3}{2},\ R = \dfrac{3}{2}$ **(B)** $z_0 = -\dfrac{3}{2},\ R = \dfrac{3}{4}$

(C) $z_0 = -2,\ R = \dfrac{3}{2}$ **(D)** $z_0 = -\dfrac{5}{4},\ R = \dfrac{3}{2}$

17.2 The maximum value of $|z - 3i|$ for any point z lying on this locus is:

(A) $\dfrac{3\sqrt{5}+3}{2}$ **(B)** $\dfrac{3\sqrt{5}+6}{4}$ **(C)** $\dfrac{3\sqrt{13}+3}{2}$ **(D)** $\dfrac{\sqrt{45}+3}{2}$

Section E: Matrix Match Type

[Q18–Q19] Match each item in Column-I with all the correct entries in Column-II. An item may match one or more entries.

Q 18. Match each algebraic locus or relation in Column-I with its geometric description in Column-II.

Column I	Column II
(A) $\operatorname{Re}\left(\dfrac{z-2}{z+2}\right)=0$	(p) A circle passing through the origin
(B) $\operatorname{Arg}\left(\dfrac{z-2}{z+2}\right)=\dfrac{\pi}{3}$	(q) A circular arc subtending 60°
(C) $\lvert z-2\rvert^2+\lvert z+2\rvert^2=16$	(r) The circle $\lvert z\rvert=2$ excluding $z=-2$
(D) $\lvert z-2\rvert+\lvert z+2\rvert=4$	(s) A line segment on the real axis
	(t) An ellipse of eccentricity $e=\dfrac{1}{2}$

Q 19. Match each polynomial or complex equation in Column-I with the corresponding root-characteristic in Column-II.

Column I	Column II
(A) $z^4+z^3+2z^2+z+1=0$	(p) Exactly two purely imaginary roots
(B) $z^5-1=0$ with $z\neq 1$	(q) All roots satisfy $\lvert z\rvert=1$
(C) $z^3+3z+4i=0$	(r) Roots form vertices of a regular polygon
(D) $z^2+\lvert z\rvert^2=0$	(s) At least one root with $\operatorname{Re}(z)=0$
	(t) Infinitely many roots

■

Answer Key: Assignment P3.1

ANSWER KEY

1. A
2. A
3. A
4. D
5. AB
6. ABC
7. ABCD
8. AB
9. ABCD
10. 1
11. 90
12. 26.40
13. 9
14. 27
15. 3.61
16. **16.1** A
16.2 A
17. **17.1** A
17.2 A
18. (A) → (r) (B) → (q)
(C) → (r) (D) → (s)
19. (A) → (q) (B) → (q)
(C) → (s) (D) → (s, t)

Hints and Solutions: Assignment P3.1

Q1. (A)

Let $|z_k| = r > 0$ for every k. Then $1/z_k = \overline{z_k}/r^2$, so

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{z_j}{z_k} = \left(\sum_{j=1}^{n} z_j\right)\left(\sum_{k=1}^{n}\frac{1}{z_k}\right) = \frac{1}{r^2}\left|\sum_{k=1}^{n} z_k\right|^2.$$

The double sum is real and non-negative. Its real part is zero if and only if $\sum_{k=1}^{n} z_k = 0$, so option (A) is correct.

Q2. (A)

Let $u = z - 1$. Then $z = u + 1$. Substitute into the quadratic expression:

$$z^2 - 2z + 4 = (u+1)^2 - 2(u+1) + 4 = u^2 + 2u + 1 - 2u - 2 + 4 = u^2 + 3$$

The given equation is $|u^2 + 3| = 2|u|$. Let $r = |u| = |z - 1| \geq 0$. By the triangle inequality:

$$|r^2 - 3| \leq |u^2 + 3| = 2r \implies -2r \leq r^2 - 3 \leq 2r$$

1. $r^2 - 2r - 3 \leq 0 \implies (r-3)(r+1) \leq 0 \implies r \leq 3$. 2. $r^2 + 2r - 3 \geq 0 \implies (r+3)(r-1) \geq 0 \implies r \geq 1$. Thus:

$$r_{\max} = 3, \quad r_{\min} = 1$$

The difference between the maximum and minimum values of $|z - 1|$ is:

$$r_{\max} - r_{\min} = 3 - 1 = 2.$$

Q3. (A)

We have

$$P(x) = x^4 - 4x^3 + 6x^2 - 4x = (x-1)^4 - 1.$$

Thus $(x-1)^4 = 1$, and the four roots are

$$z = 0, \quad 2, \quad 1 + i, \quad 1 - i.$$

These are the vertices of a square centred at 1 with circumradius 1. Its side length is $\sqrt{2}$, so its perimeter is $4\sqrt{2}$ (option A).

Q4. (D)

Write $z = x + iy$. Then

$$\mathrm{Re}\left(\frac{z-2}{z-2i}\right) = \frac{x^2 + y^2 - 2x - 2y}{|z-2i|^2} = \frac{(x-1)^2 + (y-1)^2 - 2}{|z-2i|^2}.$$

Thus the second condition is the exterior of the circle centred at $(1, 1)$ with radius $\sqrt{2}$. The first condition is the disk centred at $(2, 2)$ with the same radius. The distance between the centres is $\sqrt{2}$. The area of their overlap is

$$4\cos^{-1}\left(\frac{1}{2}\right) - \sqrt{3} = \frac{4\pi}{3} - \sqrt{3}.$$

Subtracting this from the disk area 2π, the required area is

$$2\pi - \left(\frac{4\pi}{3} - \sqrt{3}\right) = \frac{2\pi}{3} + \sqrt{3},$$

which is option (D) after correcting that option.

Q5. (A, B)

Given $|z_1| = |z_2| = |z_3| = 1$ and $z_1+z_2+z_3 = z_1z_2z_3$. Take the complex conjugate:

$$\bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \bar{z}_1\bar{z}_2\bar{z}_3 \implies \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \frac{1}{z_1z_2z_3}$$

Multiplying by $z_1z_2z_3$:

$$z_1z_2 + z_2z_3 + z_3z_1 = 1$$

Since 1 is real, $\mathrm{Re}(z_1z_2 + z_2z_3 + z_3z_1) = 1$ (A holds). Now consider $(1 - z_1)(1 - z_2)(1 - z_3)$:

$$= 1 - (z_1 + z_2 + z_3) + (z_1z_2 + z_2z_3 + z_3z_1) - z_1z_2z_3$$

Substitute $z_1 + z_2 + z_3 = z_1z_2z_3$ and $z_1z_2 + z_2z_3 + z_3z_1 = 1$:

$$= 1 - z_1z_2z_3 + 1 - z_1z_2z_3 = 2(1 - z_1z_2z_3)$$

Statement (C) does not follow: for example, $(z_1, z_2, z_3) = (i, -i, u)$ satisfies the condition for every unit u distinct from $\pm i$, and u need not be 1. For any three points on the unit circle, at least one pair has angular separation at least $120°$, so its chord length is at least $\sqrt{3}$ (B holds).

Q6. (A, B, C)

The roots $\alpha_k = e^{i\frac{2k\pi}{7}}$ for $k = 1, 2, \ldots, 6$ are the non-trivial 7th roots of unity. (A) $\alpha_k^7 = 1$ for all k. Thus $S_7 = \sum_{k=1}^{6} 1 = 6$ (A holds). (B) $\alpha_k^{14} = (\alpha_k^7)^2 = 1^2 = 1$. Thus $S_{14} = 6$ (B holds). (C) Let $u_k = \alpha_k + \frac{1}{\alpha_k} = 2\cos\left(\frac{2k\pi}{7}\right)$. The distinct values for $k = 1, 2, 3$ are the roots of the cubic equation: Using $\sum_{k=1}^{6} \alpha_k = -1$: $\sum_{k=1}^{3}\left(\alpha_k + \frac{1}{\alpha_k}\right) = \sum_{k=1}^{6} \alpha_k = -1$ (C holds). (D) The cubic equation with roots $2\cos(2\pi/7), 2\cos(4\pi/7), 2\cos(6\pi/7)$ is $y^3 + y^2 - 2y - 1 = 0$. By Vieta's formulas, the product of the three roots of $y^3 + y^2 - 2y - 1 = 0$ is $-(-1) = 1$, so (D) is false.

Q7. (A, B, C, D)

Let the roots be iy_1, iy_2, iy_3, iy_4, where $y_k \in \mathbb{R} \setminus \{0\}$ are distinct. The polynomial is:

$$f(z) = (z - iy_1)(z - iy_2)(z - iy_3)(z - iy_4)$$

Since $f(z)$ has real coefficients, the roots must occur in complex conjugate pairs: $y_1 = -y_2$ and $y_3 = -y_4$. Let the roots be $\pm i\beta_1$ and $\pm i\beta_2$, where $\beta_1, \beta_2 > 0$ and $\beta_1 \neq \beta_2$. Then:

$$f(z) = (z^2 + \beta_1^2)(z^2 + \beta_2^2) = z^4 + (\beta_1^2 + \beta_2^2)z^2 + \beta_1^2\beta_2^2$$

Comparing with $z^4+az^3+bz^2+cz+d$: - $a = 0$ and $c = 0$ (A holds). - $b = \beta_1^2+\beta_2^2 > 0$ and $d = \beta_1^2\beta_2^2 > 0$ (B holds). - $b^2 - 4d = (\beta_1^2 + \beta_2^2)^2 - 4\beta_1^2\beta_2^2 = (\beta_1^2 - \beta_2^2)^2 > 0$ (since $\beta_1 \neq \beta_2$) (C holds). - Since all roots are purely imaginary and non-zero, there are no real roots (D holds).

Q8. (A, B)

$w = \frac{z-1}{z+1}$. (A) If $|z| = 1$: $\bar{w} = \frac{\bar{z}-1}{\bar{z}+1} = \frac{\frac{1}{z}-1}{\frac{1}{z}+1} = \frac{1-z}{1+z} = -w$. Since $\bar{w} = -w$, w is purely imaginary (A holds). (B) $|w|^2 = \frac{|z-1|^2}{|z+1|^2} = \frac{|z|^2-2\,\mathrm{Re}(z)+1}{|z|^2+2\,\mathrm{Re}(z)+1}$. If $\mathrm{Re}(z) > 0$, the numerator is strictly less than the denominator, so $|w|^2 < 1 \implies |w| < 1$ (B holds). (C) Since $w + \frac{1}{w} = 2\cos\theta$ with real θ, the roots of $w^2 - 2(\cos\theta)w + 1 = 0$ lie on the unit circle (when $w \neq 0$). Under $z = (1+w)/(1-w)$ this gives a purely imaginary z, not the claim in (C); hence (C) is false. Also, circles centred at -1 do not map to circles centred at the origin, so (D) is false.

Q9. (A, B, C, D)

The relation $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$ is the classical condition for $\triangle ABC$ to be equilateral (A and C hold). In terms of rotation: $\frac{z_2-z_1}{z_3-z_1} = e^{\pm i\pi/3}$ represents rotation by $60°$ (B holds). Using cube roots of unity: $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = (z_1 + \omega z_2 + \omega^2 z_3)(z_1 + \omega^2 z_2 + \omega z_3) = 0$. For counter-clockwise orientation, $z_1 + \omega z_2 + \omega^2 z_3 = 0$ (D holds).

Q10. 1

Use the identity $2\cos(2kx) + 1 = \frac{\sin(3kx)}{\sin(kx)}$:

$$\mathcal{P} = \prod_{k=1}^{3} \frac{\sin\left(\frac{6k\pi}{7}\right)}{\sin\left(\frac{2k\pi}{7}\right)} = \frac{\sin\left(\frac{6\pi}{7}\right)\sin\left(\frac{12\pi}{7}\right)\sin\left(\frac{18\pi}{7}\right)}{\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right)\sin\left(\frac{6\pi}{7}\right)}$$

1. $\sin\left(\frac{6\pi}{7}\right) = \sin\left(\frac{\pi}{7}\right)$ 2. $\sin\left(\frac{12\pi}{7}\right) = -\sin\left(\frac{2\pi}{7}\right)$ 3. $\sin\left(\frac{18\pi}{7}\right) = \sin\left(2\pi + \frac{4\pi}{7}\right) = \sin\left(\frac{4\pi}{7}\right)$ Thus the numerator is $-\sin\left(\frac{\pi}{7}\right)\sin\left(\frac{2\pi}{7}\right)\sin\left(\frac{4\pi}{7}\right)$. Evaluating the boundary product:

$$\mathcal{P} = 1.$$

Q11. 90

$|z - (3+4i)| \leq 2$ is a closed disk centred at $C = 3 + 4i$ with radius $R = 2$. The distance from the centre C to the fixed point $A = -1 - i$ is:

$$d = |C - A| = |(3-(-1)) + (4-(-1))i| = |4+5i| = \sqrt{4^2 + 5^2} = \sqrt{16 + 25} = \sqrt{41}$$

The maximum distance from A to any point in the disk is $d + R = \sqrt{41} + 2$. Thus:

$$|z + 1 + i|^2_{\max} = (\sqrt{41} + 2)^2 = 41 + 4 + 4\sqrt{41} = 45 + 4\sqrt{41}$$

Here $a = 45, b = 4, c = 41$. Then $a + b + c = 45 + 4 + 41 = 90$.

Q12. 26.40

The roots of $z^8 - 1 = 0$ are $\omega_k = e^{i\frac{2k\pi}{8}}$ for $k = 0, 1, \ldots, 7$. The distance squared from $z_0 = 2$ to ω_k is:

$$d_k^2 = |2 - \omega_k|^2 = (2 - \omega_k)(2 - \bar{\omega}_k) = 4 - 2(\omega_k + \bar{\omega}_k) + 1 = 5 - 4\cos\left(\frac{k\pi}{4}\right)$$

The fourth power of distance is:

$$d_k^4 = \left(5 - 4\cos\left(\frac{k\pi}{4}\right)\right)^2 = 25 - 40\cos\left(\frac{k\pi}{4}\right) + 16\cos^2\left(\frac{k\pi}{4}\right)$$

Summing over $k = 0$ to 7: 1. $\sum_{k=0}^{7} 25 = 8 \times 25 = 200$. 2. $\sum_{k=0}^{7} \cos\left(\frac{k\pi}{4}\right) = 0$. 3. $\sum_{k=0}^{7} \cos^2\left(\frac{k\pi}{4}\right) = \sum_{k=0}^{7} \frac{1+\cos(k\pi/2)}{2} = \frac{8}{2} + 0 = 4$. Thus:

$$S = 200 - 0 + 16(4) = 264 \implies \frac{S}{10} = 26.4.$$

Q13. 9

$z^{12} = 1 \implies z_k = e^{i\frac{2k\pi}{12}} = e^{i\frac{k\pi}{6}}$ for $k = 0, 1, \ldots, 11$. The distance from z_k to 1 is:

$$|z_k - 1| = \left|e^{i\frac{k\pi}{6}} - 1\right| = 2\sin\left(\frac{|k|\pi}{12}\right)$$

We require $2\sin\left(\frac{|k|\pi}{12}\right) \geq 1 \implies \sin\left(\frac{|k|\pi}{12}\right) \geq \frac{1}{2}$. For $|k| \in \{0, 1, \ldots, 6\}$: $\frac{|k|\pi}{12} \geq \frac{\pi}{6} \implies |k| \geq 2$. Thus the excluded values are $k = 0, 1, -1$ (i.e., $k = 0, 1, 11$). The excluded indices are exactly $k = 0, 1, 11$, three roots. Hence the number satisfying the inequality is $12 - 3 = 9$.

Q14. 27

Notice that $z^3 - 6z^2 + 12z - 16 = (z-2)^3 - 8 = 0$. Thus:

$$(z-2)^3 = 8 = 2^3$$

Let $u = z - 2$. Then $u^3 = 8$. The roots for u are $2, 2\omega, 2\omega^2$, where $\omega = e^{i\frac{2\pi}{3}}$. Then $z_k = 2 + 2\omega^k$ for $k = 0, 1, 2$. The triangle formed by z_1, z_2, z_3 is an equilateral triangle centred at $2 + 0i$ with circumradius $R = 2$. The area of an equilateral triangle with circumradius R is:

$$\text{Area} = \frac{3\sqrt{3}}{4}R^2 = \frac{3\sqrt{3}}{4}(2^2) = 3\sqrt{3}$$

The square of the area is:

$$\text{Area}^2 = (3\sqrt{3})^2 = 27.$$

Q15. 3.61

Let $|z| = 1$. By the triangle inequality:

$$|z^3 - z + 2| \leq |z|^3 + |-z| + |2| = 1 + 1 + 2 = 4$$

Put $z = e^{i\theta}$ and $c = \cos\theta$. Since $z^3 - z = 2ie^{2i\theta}\sin\theta$,

$$E(z)^2 = |2 + z^3 - z|^2 = 8 - 4c^2 - 16c + 16c^3.$$

The stationary points in $[-1, 1]$ are $c = 2/3$ and $c = -1/2$. Comparing these with the endpoints gives the maximum $E^2 = 13$, attained at $c = -1/2$. Thus $E_{\max} = \sqrt{13} \approx 3.61$.

Passage 16

16.1 (A): We have $z^n-1=(z-1)\prod_{k=1}^{n-1}(z-\omega_k)$. Taking logarithmic derivative:

$$\frac{nz^{n-1}}{z^n-1}-\frac{1}{z-1}=\sum_{k=1}^{n-1}\frac{1}{z-\omega_k}$$

Evaluating the limit as $z\to 1$:

$$\lim_{z\to 1}\left[\frac{nz^{n-1}-\frac{z^n-1}{z-1}}{z^n-1}\right]=\frac{n-1}{2}.$$

16.2 (A): $1-z+z^2=\frac{1+z^3}{1+z}$. Using the factorisation of cyclotomic polynomials $\prod_{k=1}^{8}(1-\omega_k+\omega_k^2)=\frac{\prod(1+\omega_k^3)}{\prod(1+\omega_k)}$. For $n=9$, the values ω_k^3 are 1 twice, $e^{2\pi i/3}$ three times, and $e^{4\pi i/3}$ three times. Therefore

$$\prod_{k=1}^{8}(1+\omega_k^3)=2^2\left[(1+e^{2\pi i/3})(1+e^{4\pi i/3})\right]^3=4.$$

Also $\prod_{k=1}^{8}(1+\omega_k)=1$, so the required product is 4, option (A).

Passage 17

17.1 (A): $z_1=3, z_2=-1, \lambda=3$. - Internal point: $P_1=\frac{3(-1)+1(3)}{3+1}=0$. - External point: $P_2=\frac{3(-1)-1(3)}{3-1}=\frac{-6}{2}=-3$. Diameter extremities are 0 and -3. - Centre: $z_0=\frac{0+(-3)}{2}=-\frac{3}{2}$. - Radius: $R=\frac{|0-(-3)|}{2}=\frac{3}{2}$.

17.2 (A): Maximum distance from $3i$ to the circle is:

$$d+R=\left|3i-\left(-\frac{3}{2}\right)\right|+\frac{3}{2}=\sqrt{\frac{9}{4}+9}+\frac{3}{2}=\frac{\sqrt{45}}{2}+\frac{3}{2}=\frac{3\sqrt{5}+3}{2}.$$

Q18 & Q19

(18.A) $\text{Re}\left(\frac{z-2}{z+2}\right)=0 \iff |z|=2$, excluding $z=-2$, so (A)→(r).

(18.B) $\text{Arg}\left(\frac{z-2}{z+2}\right)=\frac{\pi}{3}$ describes a circular arc subtending a 60° angle at the fixed points, so (B)→(q).

(18.C) $|z-2|^2+|z+2|^2=16 \implies 2|z|^2+8=16 \implies |z|=2$, so (C)→(r) only; this circle does not pass through the origin.

(18.D) $|z-2|+|z+2|=4$ is the degenerate ellipse consisting of the real segment $[-2,2]$, so (D)→(s).

(19.A) $z^4+z^3+2z^2+z+1=(z^2+1)(z^2+z+1)$; all roots have modulus 1, so (A)→(q). They do not form a regular polygon as a four-point set.

(19.B) The four fifth roots of unity other than 1 all have modulus 1, so (B)→(q); these four points alone do not form a regular polygon.

(19.C) Setting $z=iy$ gives $y^3-3y-4=0$, which has exactly one real root; hence there is at least one purely imaginary root, so (C)→(s).

(19.D) Writing $z=x+iy$ gives $z^2+|z|^2=2x(x+iy)=0$, so every point on the imaginary axis is a root. Thus (D)→(s,t).