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Assignment P6.1 with answer key and solutions

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ASSIGNMENT P6.1

Limits, Squeeze Theorem & Asymptotic Expansion

Focus Topics: *Limits by expansion and standard results, the squeeze theorem, oscillating limits, with first links to continuity, differentiability and Rolle's theorem.*

Section A: Single Correct Choice

[Q1–Q4] Each question has four options (A), (B), (C) and (D). Exactly ONE option is correct.

Q 1. Let

$$f(x)=\lim_{n\to\infty}\tan^{-1}\left(4n^2\left(1-\cos\frac{x}{n}\right)\right),\qquad g(x)=\lim_{n\to\infty}\frac{n^2}{2}\ln\left(\cos\frac{2x}{n}\right).$$

Then

$$\lim_{x\to0}\frac{e^{-2g(x)}-e^{f(x)}}{x^6}$$

is equal to:

(A) $-\frac{5}{3}$ **(B)** $\frac{5}{3}$ **(C)** $-\frac{8}{3}$ **(D)** $\frac{8}{3}$

Q 2. The number of values of $x\in[-4,4]$ at which the function

$$f(x)=\lfloor 3x+14\rfloor+|4x^2-1|(2x^2+3x-2)+\sin\frac{\pi x}{2}$$

is non-differentiable is:

(A) 23 **(B)** 24 **(C)** 25 **(D)** 26

Q 3. If

$$f(x)=\big|\,|\sin(|x|-2)|-3\,\big|,$$

then which statement is correct?

(A) f is continuous but not differentiable at $x=4$.

(B) f is discontinuous at $x=4$.

(C) f is differentiable at $x=4$ and $f'(4)=\cos 2$.

(D) f is differentiable at $x=4$ and $f'(4)=-\cos 2$.

Q 4. If $x_1 = \sqrt{3}$ and

$$x_{n+1} = \frac{x_n}{1+\sqrt{1+x_n^2}}, \qquad n \in \mathbb{N},$$

then $\lim_{n\to\infty} 2^n x_n$ is:

(A) $\frac{3}{2\pi}$ **(B)** $\frac{2}{3\pi}$ **(C)** $\frac{2\pi}{3}$ **(D)** $\frac{3\pi}{2}$

Section B: One or More than One Correct

[Q5–Q9] Each question has four options (A), (B), (C) and (D). ONE OR MORE options may be correct.

Q 5. Let $a, b \in \mathbb{Z}$ and

$$L = \lim_{x\to 0} \frac{\sin(x^2) + 2\cos(bx) - ax^4 - 2}{e^{ax} - 1 - ax - 2x^2 - \frac{a^3x^3}{6}}.$$

Identify the correct statement(s):

(A) If $L = \frac{25}{8}$, then $a+b$ can be 3.

(B) If $L = \frac{25}{8}$, then $a+b$ can be -3.

(C) If $L = -\frac{23}{8}$, then $a+b$ can be -1.

(D) If $L = -\frac{23}{8}$, then $a+b$ can be 1.

Q 6. Let $f(x) = x^2 \sin\frac{1}{x}$ for $0 < x \le 1$, with $f(0) = 0$, and let $g(x) = x^2$ on $[0,1]$. Which statement(s) is/are correct?

(A) f is differentiable on $[0,1]$.

(B) $\lim_{x\to 0} \frac{f(x)}{g(x)}$ does not exist.

(C) $\lim_{x\to 0} f'(x)g'(x)$ does not exist.

(D) $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

Q 7. If f is differentiable on $[a,b]$ and $f(a) = f(b) = 0$, then for some $x \in (a,b)$ which statement(s) must hold?

(A) $xf(x) + f'(x) = 0$.

(B) $xf(x) - f'(x) = 0$.

(C) $50f(x) + f'(x) = 0$.

(D) $f'(x) + (\ln 50)f(x) = 0$.

Q 8. Let f, g be continuous and differentiable on $\mathbb{R}$. Which statement(s) is/are correct?

(A) If $f(a) = f(b) = 0$, then for every $\beta \in \mathbb{R}$ there exists $c \in (a,b)$ such that $\beta f(c) + f'(c) = 0$.

(B) If $f(a) = f(b) = 0$, there exists $c \in (a, b)$ such that $e^c f(c) + f'(c) = 0$.

(C) If f, g are nonzero on $[a, b]$ and $f(a)g(b) = f(b)g(a)$, then some $c \in (a, b)$ satisfies $\dfrac{f'(c)}{f(c)} = \dfrac{g'(c)}{g(c)}$.

(D) If $f(b)^2 - f(a)^2 = b^2 - a^2$, then $f'(x)f(x) - x = 0$ has no root in (a, b).

Q 9. Let $P(x)$ be a polynomial of degree n and define

$$f(x) = \begin{cases} P\left(\dfrac{1}{x^3}\right) e^{-1/x^4}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Which statement(s) is/are correct?

(A) f is discontinuous at $x = 0$.

(B) f is continuous at $x = 0$.

(C) f is non-differentiable at $x = 0$.

(D) $f'(0) = \lim\limits_{x \to 0} f(x)$.

Section C: Numerical Answer Type

[Q10–Q15] The answer to each question is a numerical value: an integer, or a decimal rounded off to two decimal places.

Q 10. For $p, n \in \mathbb{N}$, let $f(x) = 1 - x^p$ and

$$g_n(x) = \frac{n}{\dfrac{1}{f(x)} + \dfrac{1}{f(2x)} + \cdots + \dfrac{1}{f(nx)}}.$$

Find

$$\lim_{x \to 0} \frac{1 - g_n(x)}{x^p} \quad \text{for } n = 5,\ p = 3.$$

Q 11. Evaluate $\pi^2 k$, where

$$k = \lim_{x \to 0} \frac{\tan^3 x - x^3 \operatorname{sgn}\left(\left\lfloor \dfrac{x}{\tan^{-1} x} \right\rfloor\right)}{x \sin^2 x (1 - \cos \pi x)}.$$

Here $\lfloor \cdot \rfloor$ denotes the greatest integer function._____

Q 12. For $k \in \mathbb{N}$, let

$$g(k) = \lim_{n \to \infty} n^k \int_0^{1/n} x^{k-1}\, dx.$$

Find $\left\lfloor \dfrac{1}{g(9)} \right\rfloor$._____

Q 13. Let

$$S_n = \sum_{r=1}^{n} \tan^{-1}\left(\frac{2(2r-1)}{4 + r^2(r-1)^2}\right).$$

Evaluate

$$\lim_{n\to\infty} \sum_{r=2}^{n} \big(\cot S_{r-1} - \cot S_r\big).$$

Q 14. If

$$\lim_{n\to\infty} \left(\prod_{r=1}^{n} \sin\frac{r\pi}{4n} \cos\frac{r\pi}{4n}\right)^{1/n} = \frac{a}{b}, \qquad a, b \in \mathbb{N},$$

find the least value of $a + b$._____

Q 15. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, with f' continuous on $\mathbb{R}$ and $f'(3) = 18$. Define

$$g_n(x) = n\left[f\left(x + \frac{5}{n}\right) - f\left(x - \frac{2}{n}\right)\right].$$

Find $\lim_{n\to\infty} g_n(3)$._____

Section D: Linked Comprehension Type

[Q16] The passage is followed by two linked numerical-answer questions.

Q 16. Passage: Rolle's Theorem for a Cubic

Let $f(x)$ be a cubic polynomial with leading coefficient 1 such that $f(a) = b$ and $f'(a) = f''(a) = 0$. Let

$$g(x) = f(x) - f(a) + (a - x)f'(x) + 3(x - a)^2,$$

and suppose the conclusion of Rolle's theorem for g on $[a, b]$ holds at $x = 2$, where $2 \in (a, b)$.

Answer the following:

16.1 The value of $f''(2)$ is _____.

16.2 The value of $64\int_a^b f(x)\,dx$ is _____.

Section E: Matrix Match Type

[Q18–Q19] Match each item in Column-I with its corresponding entry in Column-II.

Q 18. Match each item in Column-I with its value in Column-II.

Column I	Column II
(A) If $L = \lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/(1-\cos x)}$ and $\ln L = k$, then $\dfrac{1}{k^2}$ is	(p) 0
(B) $\lim\limits_{n\to\infty}\left(\dfrac{n^2}{n-1}\right)^{\tan^{-1}(1/\sqrt{n})}$ is	(q) 1
(C) The number of solutions of $8\sin^4 x + 8\cos^4 x = 5$ in $0 < x < 2\pi$ is	(r) 3
(D) If f is a non-constant polynomial and $g(x) = \lvert x(x-1)(x-2)f(x)\rvert$ is differentiable for all $x \in \mathbb{R}$, the minimum number of distinct roots of $f(x) = 0$ is	(s) 8
	(t) 9

Q 19. Match each item in Column-I with its value in Column-II.

Column I	Column II
(A) $\lim\limits_{x\to 0}\dfrac{1-\cos 2x}{e^{x^2}-e^x+x}$	(p) 1
(B) If $\lim\limits_{x\to 0^+}\left(\dfrac{(3/x)+1}{(3/x)-1}\right)^{1/x} = e^{p/q}$, where p, q are coprime positive integers, then $p+q$ is	(q) 2
(C) $\lim\limits_{x\to 0}\dfrac{\tan^3 x - \tan(x^3)}{x^5}$	(r) 4
(D) $\lim\limits_{x\to 0}\dfrac{x+2\sin x}{\sqrt{x^2+2\sin x+1}-\sqrt{\sin^2 x - x + 1}}$	(s) 5

■

Answer Key: Assignment P6.1

ANSWER KEY

1. D

2. C

3. D

4. C

5. BD

6. ABD

7. ABCD

8. ABC

9. BD

10. 45

11. 2

12. 9

13. 2

14. 5

15. 126

16. **16.1** 6

16.2 321

18. (A) → (t) (B) → (q)
(C) → (s) (D) → (r)

19. (A) → (r) (B) → (s)
(C) → (p) (D) → (q)

Hints and Solutions: Assignment P6.1

Q1. (D)

The two limits defining the auxiliary functions simplify to $f(x) = \tan^{-1}(2x^2)$ and $g(x) = -x^2$. Thus the numerator is $e^{2x^2} - e^{\tan^{-1}(2x^2)}$. Using $\tan^{-1}(2x^2) = 2x^2 - \frac{8}{3}x^6 + O(x^{10})$ gives the limit $8/3$.

Q2. (C)

On $[-4, 4]$, $\lfloor 3x + 14 \rfloor$ jumps wherever $3x$ is an integer. In the open interval this gives $3x \in \{-11, \ldots, 11\}$, i.e. 23 points; at the right end $x = 4$ the function also jumps (left limit 25, value 26), while at $x = -4$ it is constant on the right. The term $|4x^2 - 1|(2x - 1)(x + 2)$ has a cusp at $x = -\frac{1}{2}$ only (at $x = \frac{1}{2}$ the factor $(2x - 1)$ cancels the cusp), and $-\frac{1}{2}$ is not a jump point of the floor term. $\sin\frac{\pi x}{2}$ is smooth. Total: $23 + 1 + 1 = 25$.

Q3. (D)

Near $x = 4$, $|x| = x$ and $\sin(x - 2) > 0$. Therefore $f(x) = 3 - \sin(x - 2)$ locally, so $f'(4) = -\cos 2$; option D.

Q4. (C)

Put $x_n = \tan\theta_n$. Then $\tan\theta_{n+1} = \tan(\theta_n/2)$, so $\theta_n = \pi/(3\cdot 2^{n-1})$. Since $\tan t \sim t$, $2^n x_n \to 2\pi/3$.

Q5. (B, D)

Expanding, the numerator is $(1-b^2)x^2 + \left(\frac{b^4}{12} - a\right)x^4 + O(x^6)$ and the denominator is $\left(\frac{a^2}{2} - 2\right)x^2 + \frac{a^4}{24}x^4 + O(x^5)$. For $L = \frac{25}{8}$ or $-\frac{23}{8}$ with integers a, b, the x^2 terms must vanish: $b^2 = 1$, $a^2 = 4$. Then $L = \dfrac{\frac{1}{12} - a}{2/3}$, which equals $\frac{25}{8}$ for $a = -2$ and $-\frac{23}{8}$ for $a = 2$. With $b = \pm 1$: $L = \frac{25}{8} \Rightarrow a + b \in \{-1, -3\}$ and $L = -\frac{23}{8} \Rightarrow a + b \in \{1, 3\}$. Hence (B) and (D).

Q6. (A, B, D)

At 0, $f'(0) = \lim_{x\to 0} x\sin(1/x) = 0$, so f is differentiable. The ratio $f/g = \sin(1/x)$ has no limit, while $f'g' = 4x^2\sin(1/x) - 2x\cos(1/x) \to 0$. The ratio f'/g' has no limit. Hence A, B and D are correct.

Q7. (A, B, C, D)

Apply Rolle's theorem to $e^{x^2/2}f(x)$, $e^{-x^2/2}f(x)$, $e^{50x}f(x)$, and $e^{(\ln 50)x}f(x)$, respectively. Each auxiliary function vanishes at both endpoints, giving A–D.

Q8. (A, B, C)

Apply Rolle's theorem to $e^{\beta x}f(x)$ for A and to $e^{e^x}f(x)$ for B. For C, apply Rolle's theorem to f/g; D is false because Rolle's theorem applied to $f(x)^2 - x^2$ guarantees a root of $f'f - x$ in (a, b). Thus ABC.

Q9. (B, D)

For every polynomial P, the factor e^{-1/x^4} tends to zero faster than any power of $1/x$ grows. Hence $f(x) \to 0 = f(0)$ and $\lim_{x\to 0} f(x)/x = 0$, so f is continuous and differentiable at 0 with $f'(0) = 0$. Thus B and D.

Q10. 45

For $p=3$, $1/f(kx)=1+k^3x^3+O(x^6)$. Hence $g_5(x)=1-\frac{1+8+27+64+125}{5}x^3+O(x^6)=1-45x^3+O(x^6)$, giving 45.

Q11. 2

Since $x/\tan^{-1}x>1$ for small nonzero x, its floor is 1 and the signum is 1. Also $\tan^3 x-x^3=x^5+O(x^7)$ and $x\sin^2 x(1-\cos\pi x)\sim(\pi^2/2)x^5$. Thus $\pi^2 k=2$.

Q12. 9

Direct integration gives $g(k)=\lim_{n\to\infty} n^k\frac{(1/n)^k}{k}=1/k$. Therefore $\lfloor 1/g(9)\rfloor=9$.

Q13. 2

With $u_r=\frac{r^2}{2}$, $u_{r-1}=\frac{(r-1)^2}{2}$: $\dfrac{u_r-u_{r-1}}{1+u_ru_{r-1}}=\dfrac{(2r-1)/2}{1+r^2(r-1)^2/4}=\dfrac{2(2r-1)}{4+r^2(r-1)^2}$.
So $S_n=\tan^{-1}\frac{n^2}{2}$ and $\cot S_n=\frac{2}{n^2}$. The sum telescopes to $\cot S_1-\cot S_n=2-\frac{2}{n^2}\to 2$.

Q14. 5

Use $\sin t\cos t=\frac{1}{2}\sin 2t$ and the product identity $\prod_{r=1}^{n}\sin\frac{r\pi}{2n}=\frac{\sqrt{n}}{2^{n-1}}$. The product inside the nth root is $\sqrt{n}/2^{2n-1}$, so the limit is $1/4$ and the least $a+b$ is 5.

Q15. 126

By differentiability, $f(3+5/n)-f(3-2/n)=(7/n)f'(3)+o(1/n)$. Multiplication by n gives $7\cdot 18=126$.

Passage 16

16.1: Since $f'(a)=f''(a)=0$ and f is a monic cubic, $f(x)=(x-a)^3+b$. Then $g'(x)=(a-x)f''(x)+6(x-a)=(x-a)\,(6-6(x-a))$; $g'(2)=0$ with $2\neq a$ gives $a=1$, hence $f''(2)=6(2-1)=6$.
16.2: $g(a)=0$ and $g(b)=d^3-3d^3+3d^2$ with $d=b-a$; $g(b)=0$ gives $d=\frac{3}{2}$, so $b=\frac{5}{2}$. Then $64\displaystyle\int_1^{5/2}\left((x-1)^3+\tfrac{5}{2}\right)dx=64\left(\tfrac{81}{64}+\tfrac{15}{4}\right)=321$.

Q18

(A) $\ln L=\lim\dfrac{\ln(\sin x/x)}{1-\cos x}=\dfrac{-x^2/6}{x^2/2}=-\frac{1}{3}$, so $1/k^2=9$: (A) $\to$ (t). (B) The logarithm is $\tan^{-1}\frac{1}{\sqrt{n}}\cdot\ln\frac{n^2}{n-1}\sim\frac{\ln n}{\sqrt{n}}\to 0$, so the limit is 1: (B) $\to$ (q). (C) $8(1-2\sin^2 x\cos^2 x)=5$ gives $\sin^2 2x=\frac{3}{4}$, which has 8 solutions in $(0,2\pi)$: (C) $\to$ (s). (D) $|h|$ is differentiable at a simple zero of h only if that zero has even multiplicity, so f must vanish at $0,1,2$: (D) $\to$ (r).

Q19

(A) $1-\cos 2x\sim 2x^2$ and $e^{x^2}-e^x+x=\frac{x^2}{2}+O(x^3)$, so the limit is 4: (A) $\to$ (r). (B) $\frac{1}{x}\ln\frac{3+x}{3-x}\to\frac{2}{3}$, so $p+q=5$: (B) $\to$ (s). (C) $\tan^3 x=x^3+x^5+\ldots$, $\tan x^3=x^3+O(x^9)$, limit 1: (C) $\to$ (p). (D) The difference of radicands is $x^2+2\sin x-\sin^2 x+x\sim 3x$; dividing by the sum $\to 2$ gives $\frac{3x}{2}$, and the limit is $\frac{3x}{3x/2}=2$: (D) $\to$ (q).