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ASSIGNMENT P7.1

Indefinite Integration: Substitution, Parts & Reduction Formulae

Focus Topics: *Trigonometric, Euler-type and reciprocal substitutions, integration by parts and the $e^x\{f(x)+f'(x)\}$ form, partial fractions, reduction formulae, with a passage on the beta function.*

Section A: Single Correct Choice

[Q1–Q4] Each question has four options (A), (B), (C) and (D). Exactly ONE option is correct.

Q 1. If $I_n = \int \dfrac{dx}{(x^2+a^2)^n}$, then $I_{n+1} + \dfrac{1-2n}{2n}\cdot\dfrac{1}{a^2}I_n$ is equal to:

(A) $\dfrac{x}{(x^2+a^2)^n}$ **(B)** $\dfrac{1}{2na^2}\cdot\dfrac{1}{(x^2+a^2)^{n-1}}$

(C) $\dfrac{1}{2na^2}\cdot\dfrac{1}{(x^2+a^2)^n}$ **(D)** $\dfrac{1}{2na^2}\cdot\dfrac{x}{(x^2+a^2)^n}$

Q 2. If $I_{n,-m} = \int \dfrac{\sin^n x}{\cos^m x}\,dx$, then $I_{n,-m} + \dfrac{n-1}{m-1}I_{n-2,2-m}$ is equal to:

(A) $\dfrac{\sin^{n-1}x}{\cos^{m-1}x}$ **(B)** $\dfrac{1}{m-1}\cdot\dfrac{\sin^{n-1}x}{\cos^{m-1}x}$

(C) $\dfrac{1}{n-1}\cdot\dfrac{\sin^{n-1}x}{\cos^{m-1}x}$ **(D)** $\dfrac{n-1}{m-1}\cdot\dfrac{\sin^{n-1}x}{\cos^{m-1}x}$

Q 3. On making the appropriate substitution from Column 2, the integral in Column 1 reduces to an integral in Column 3.

Column 1	**Column 2**	**Column 3**
(A) $\int \dfrac{2\,dx}{\sin 3x + 3\sin x}$	(P) $\sin x = t$	(I) $\int \dfrac{dt}{(2t^2-1)(1+t^2)}$
(B) $\int \dfrac{2\sin x + \sin 2x}{6\sin x - \sin 2x}\,dx$	(Q) $\cos x = t$	(II) $\int \dfrac{dt}{(2t^2+1)(1+t^2)}$
(C) $\int \dfrac{4\sin x}{\sin 4x}\,dx$	(R) $\tan x = t$	(III) $\int \dfrac{dt}{(1-t^2)(1-2t^2)}$
(D) $\int \dfrac{1-\sin^2 x}{\sin^2 x - \cos 2x}\,dx$	(S) $\tan\dfrac{x}{2} = t$	(IV) $\int \dfrac{-dt}{(1-t^2)(1+2t^2)}$
		(V) $\int \dfrac{2\,dt}{(2t^2+1)(1+t^2)}$

Select the CORRECT combination.

(A) A–P–III **(B)** B–S–V **(C)** C–P–IV **(D)** D–S–II

Q 4. Let $I_n = \int \frac{(x^2+1)^n}{x^{n+1}}\,dx$. If $f(x) = nI_n - 4(n-1)I_{n-2}$ for $n \geq 2$, with $f(1) = 0$, then for the values of n in Column 1, match Columns 2 and 3.

Column 1	Column 2	Column 3
(A) $n = 2$	(P) $f(-1) =$	(I) -1
(B) $n = 3$	(Q) $\lim\limits_{x\to 0} x^{n+1} f'(x) =$	(II) 1
(C) $n = 4$	(R) $\lim\limits_{x\to 0} x^n f(x) =$	(III) n
(D) $n = 5$	(S) $\lim\limits_{x\to\infty} \frac{f(x)}{x^n} =$	(IV) 0

Select the CORRECT combination.

(A) A–S–I **(B)** B–R–II **(C)** C–P–IV **(D)** D–Q–II

Section B: One or More than One Correct

[Q5–Q9] Each question has four options (A), (B), (C) and (D). ONE OR MORE options may be correct.

Q 5. If

$$\int \frac{(1+x)\left((1-x+x^2)(1+x+x^2)+x^2\right)}{1+2x+3x^2+4x^3+3x^4+2x^5+x^6}\,dx = g(x) + c$$

and $g(1) = \ln 2$, then which of the following is/are true? (c is the constant of integration and $[\cdot]$ is the greatest integer function.)

(A) $g(x)$ is an increasing function for all $x > -1$.

(B) $[g(99) - g(3)] = 3$.

(C) $g(x)$ is a periodic function.

(D) $g(6) - g(7) > 0$.

Q 6. If $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x}\,dx = \ln\big(f(x)\big) + g(x) + c$, where c is the constant of integration and $f(x)$ is positive, with $f'(0) = 2$, $f(0) = 1$ and $g(0) = 0$, then which of the following is/are correct?

(A) $f(x) + g(x)$ is an even function.

(B) $f(x) - g(x)$ is an increasing function.

(C) $f(x) - g(x)$ is neither increasing nor decreasing.

(D) If $h(x) = e^x - f(x) - g(x)$, then the maximum value of $h(x)$ is 1.

Q 7. If

$$\int \frac{\left(\frac{1}{x}+\frac{1}{x^2}\right)(x-1)\,dx}{\left(\frac{1}{x^4}+\frac{1}{x^2}\right)\sqrt{(x^4-x^3+x^2)(x^4+x^3+x^2)}} = \sec^{-1}|f(x)|+c,$$

then:

(A) the local maximum value of $f(x)$ is -2

(B) the local minimum value of $f(x)$ is 2

(C) the range of $f(x)$ is $\mathbb{R}\setminus\{0\}$

(D) $f(\pi) < f(e)$

Q 8. If $\int \frac{x^2-x+1}{(x^2+1)^{3/2}}e^x\,dx = e^x f(x)+c$ and $f(0)=1$, then which of the following is/are true?

(A) $f(x)$ is an even function

(B) $f(x)$ is a bounded function

(C) the range of $f(x)$ is $(0,1]$

(D) $f(x)$ has two points of maxima

Q 9. Let $y^2 = 3x^2+2x+1$ and $I_n = \int \frac{x^n}{y}\,dx$. If $aI_{10}+bI_9+cI_8 = x^9 y$, then:

(A) $a+b=49$

(B) $b+c=28$

(C) $a-b+c=20$

(D) $b^2-c^2=280$

Section C: Numerical Answer Type

[Q10–Q15] The answer to each question is a numerical value: an integer, or a decimal rounded off to two decimal places.

Q 10. If

$$\int \frac{(e^x-1)\sin x-(e^x-1-x)\cos x}{1+(e^x-1-x-\cos x)(e^x-1-x+\cos x)}\,dx = f\big[g(x)\big]+c,$$

where c is the constant of integration and $\lim\limits_{x\to 0} g(x)=0$. If $\lim\limits_{x\to 0}\left(\frac{f(x)}{x}+\frac{g(x)}{x}\right) = \frac{m}{n}$, where $m,n\in\mathbb{N}$, then the minimum value of $m+n$ is _____.

Q 11. If

$$\int \frac{dx}{(\sec x+\tan x+\operatorname{cosec} x+\cot x)^2} = \frac{x}{a}+\frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{b}+\frac{\cos 2x}{c}+d,$$

then the value of $|a+b+c|$ is _____.

Q 12. If

$$\int \frac{dy}{y^2\left(1+y^2\right)^3} = \frac{P}{y} + Q\tan^{-1} y + R\sin\left(S\tan^{-1} y\right) + T\sin\left(4\tan^{-1} y\right) + C,$$

then the value of $P - \frac{8}{3}Q - 2R + S - 32T$ is _____.

Q 13. If $\int \frac{2x+3}{x(x+1)(x+2)(x+3)+1}\,dx = c - \frac{1}{f(x)}$, where $f(0) = 1$ and c is the constant of integration, then the value of $f(1)$ is _____.

Q 14. For $x \in \left(0, \frac{\pi}{2}\right)$, if

$$\int \frac{\tan x\sqrt{\sec x}\left(1+\cos^2 x\right)}{\sqrt{\cos x + \sin^2 x}}\,dx = 2\sqrt{f(x)+1} + c$$

and $f\left(\frac{\pi}{3}\right) = \frac{3}{2}$, then $\left[f\left(\frac{\pi}{4}\right)\right]$ is _____. (Here $[\cdot]$ denotes the greatest integer function.)

Q 15. If

$$\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}}\,dx = \frac{A}{\pi}\left[-\left(\sin^{-1}\sqrt{x}\right)(1-2x) + \sqrt{x}\sqrt{1-x}\right] - x + C,$$

then A is equal to _____.

Section D: Linked Comprehension Type

[Q16–Q17] Each passage is followed by two linked numerical-answer questions.

Q 16. Passage: Recovering the Integrand

Let $f(x)$ be a polynomial function of degree 2 satisfying

$$\int \frac{f(x)}{x^3-1}\,dx = \ln\left|\frac{x^2+x+1}{x-1}\right| + \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C,$$

where C is the constant of integration.

Answer the following:

16.1 The value of $f(5)$ is _____.

16.2 Let $\int \frac{1-6\operatorname{cosec} x}{6+f(\sin x)}\,d(\sin x) = g(x) + K$, where $g(x)$ contains no constant term and K is the constant of integration. Then $\lim\limits_{t\to\pi/2} g(t)$ is equal to _____.

Q 17. Passage: The Beta Function

If $m > 0$, $n > 0$, the definite integral $I = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ depends on m and n; it is denoted by $\beta(m,n)$ and called the beta function. For example,

$$\int_0^1 x^4(1-x)^5\,dx = \beta(5,6), \qquad \int_0^1 \frac{x^{5/2}}{\sqrt{1-x}}\,dx = \beta\left(\frac{7}{2},\frac{1}{2}\right).$$

Clearly $\beta(n,m) = \beta(m,n)$.

Answer the following:

17.1 If $\int_0^{\pi/2} \cos^{2m}\theta \sin^{2n}\theta\,d\theta = \frac{1}{w}\beta\left(m+\frac{1}{u},\ n+\frac{1}{v}\right)$, then $u+v+w$ equals _____.

17.2 If $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = k\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$, then k is equal to _____.

Section E: Matrix Match Type

[Q18–Q19] Match each item in Column-I with its corresponding entries in Column-II. An item may match one or more entries.

Q 18. Match the following (c is the constant of integration).

Column I	Column II
(A) $\int (\tan x)^{1/3}dx =$ $A\ln\frac{t^4-t^2+1}{(t^2+1)^2} + B\tan^{-1}\left(\frac{2t^2-1}{\sqrt{3}}\right)+c$, where $t=\tan^{1/3}x$	(p) $A=\frac{1}{4}$
(B) $\int \frac{\sin x+\sin^3 x}{\cos 2x}\,dx =$ $A\cos x + B\ln\left\lvert\frac{\sqrt{2}\cos x+1}{\sqrt{2}\cos x-1}\right\rvert + c$	(q) $A=\frac{1}{3}$
(C) $\int \frac{dx}{(x^2+1)(x^2+4)} = A\tan^{-1}x + B\tan^{-1}\left(\frac{x}{2}\right)+c$	(r) $A=\frac{1}{2}$
(D) $\int \frac{1-x^2}{(1+x^2)\sqrt{1+x^4}}\,dx = A\sqrt{2}\sin^{-1}\left(\frac{\sqrt{2}x}{x^2+1}\right)+c$	(s) $B=\frac{\sqrt{3}}{2}$
	(t) $B=\frac{3}{4\sqrt{2}}$

Q 19. Match the following (c is the constant of integration; $x > 0$ in (C)).

Column I	Column II
(A) $\int \left(\dfrac{x^2+\cos^2 x}{1+x^2}\right) \operatorname{cosec}^2 x\, dx =$ $A\cot^{-1} x + B\dfrac{\operatorname{cosec} x}{\sec x} + c$	(p) $A = 1$
(B) $\int \sqrt{x+\sqrt{x^2+2}}\, dx =$ $\dfrac{A}{3}\left(x+\sqrt{x^2+2}\right)^{3/2} - \dfrac{B}{\left(x+\sqrt{x^2+2}\right)^{1/2}} + c$	(q) $B = -1$
(C) $\int \sqrt{\dfrac{2-x-x^2}{x^2}}\, dx = A\sqrt{2-x-x^2} -$ $\dfrac{B}{\sqrt{2}} \ln\left(\dfrac{4-x+2\sqrt{2}\sqrt{2-x-x^2}}{x}\right) -$ $\dfrac{1}{2}\sin^{-1}\left(\dfrac{2x+1}{3}\right) + c$	(r) $B = 2$
(D) $\int \dfrac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx = B\cot^{-1}\left(\tan^2 x\right) + c$	(s) $A = -1$

■

Answer Key: Assignment P7.1

ANSWER KEY

1. D
2. B
3. B
4. C
5. AB
6. BD
7. AB
8. ABC
9. ABCD
10. 5
11. 4
12. 8
13. 5
14. 0
15. 2
16. 16.1 17
    16.2 1.10
17. 17.1 6
    17.2 1
18. (A) → (p, s) (B) → (r, t)
    (C) → (q) (D) → (r)
19. (A) → (p, q) (B) → (p, r)
    (C) → (p, r) (D) → (q)

Hints and Solutions: Assignment P7.1

Q1. (D)

Integrate $I_n = \int 1 \cdot \dfrac{dx}{(x^2+a^2)^n}$ by parts:

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2}{(x^2+a^2)^{n+1}}dx = \frac{x}{(x^2+a^2)^n} + 2n\left(I_n - a^2 I_{n+1}\right).$$

Hence $2na^2 I_{n+1} = \dfrac{x}{(x^2+a^2)^n} + (2n-1)I_n$, i.e. $I_{n+1} + \dfrac{1-2n}{2n}\cdot\dfrac{1}{a^2}I_n = \dfrac{1}{2na^2}\cdot\dfrac{x}{(x^2+a^2)^n}$.

Q2. (B)

Integrate by parts with $u = \sin^{n-1}x$ and $dv = \dfrac{\sin x}{\cos^m x}dx$, so $v = \dfrac{1}{(m-1)\cos^{m-1}x}$ and $du = (n-1)\sin^{n-2}x\cos x\,dx$:

$$I_{n,-m} = \frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x} - \frac{n-1}{m-1}\int \frac{\sin^{n-2}x}{\cos^{m-2}x}\,dx = \frac{\sin^{n-1}x}{(m-1)\cos^{m-1}x} - \frac{n-1}{m-1}I_{n-2,2-m}.$$

Hence $I_{n,-m} + \dfrac{n-1}{m-1}I_{n-2,\,2-m} = \dfrac{1}{m-1}\cdot\dfrac{\sin^{n-1}x}{\cos^{m-1}x}$.

Q3. (B)

(A) $\sin 3x + 3\sin x = 2\sin x\,(1+2\cos^2 x)$; with $\cos x = t$, $dx = -\frac{dt}{\sin x}$, the integral becomes $\int \dfrac{-dt}{(1-t^2)(1+2t^2)}$: A–Q–IV.

(B) $\dfrac{2\sin x + \sin 2x}{6\sin x - \sin 2x} = \dfrac{1+\cos x}{3-\cos x}$; with $\tan\frac{x}{2} = t$ this is $\dfrac{1}{1+2t^2}$ and $dx = \dfrac{2dt}{1+t^2}$: B–S–V.

(C) $\dfrac{4\sin x}{\sin 4x} = \dfrac{1}{\cos x\cos 2x}$; with $\sin x = t$, $dx = \frac{dt}{\cos x}$, giving $\int \dfrac{dt}{(1-t^2)(1-2t^2)}$: C–P–III.

(D) $\sin^2 x - \cos 2x = 2\sin^2 x - \cos^2 x$, so the integrand is $\dfrac{1}{2\tan^2 x - 1}$; with $\tan x = t$: $\int \dfrac{dt}{(2t^2-1)(1+t^2)}$: D–R–I.

Of the options, only B–S–V is correct.

Q4. (C)

Differentiating,

$$f'(x) = \frac{n(x^2+1)^n}{x^{n+1}} - \frac{4(n-1)(x^2+1)^{n-2}}{x^{n-1}} = \frac{d}{dx}\left[\frac{(x^2+1)^{n-1}(x^2-1)}{x^n}\right],$$

and $f(1) = 0$ gives

$$f(x) = \frac{(x^2+1)^{n-1}(x^2-1)}{x^n}.$$

For every $n \geq 2$: $f(-1) = 0$ (IV); $x^{n+1}f'(x) \to n$ (III); $x^n f(x) \to -1$ (I); $\dfrac{f(x)}{x^n} \to 1$ as $x \to \infty$ (II). So P–IV, Q–III, R–I, S–II, and only C–P–IV is correct.

Q5. (A, B)

The numerator is $(1+x)(x^4+2x^2+1) = (1+x)(1+x^2)^2$ and the denominator factorises as $(1+x)^2(1+x^2)^2$. So the integrand is $\dfrac{1}{1+x}$ and, with $g(1) = \ln 2$, $g(x) = \ln(1+x)$. It increases for $x > -1$ (A); $g(99) - g(3) = \ln 25 \approx 3.22$, so $[g(99) - g(3)] = 3$ (B); it is not periodic (C false), and $g(6) < g(7)$ (D false).

Q6. (B, D)

Write the numerator as $(e^x + \cos x + 1) - (e^x + \sin x + x)$. Then the integrand is $\dfrac{(e^x + \sin x + x)'}{e^x + \sin x + x} - 1$, so $f(x) = e^x + \sin x + x$ (which has $f(0) = 1$, $f'(0) = 2$) and $g(x) = -x$. $f + g = e^x + \sin x$ is not even; $f - g = e^x + \sin x + 2x$ has derivative $e^x + \cos x + 2 > 0$, so it is increasing (B, not C); $h(x) = e^x - f - g = -\sin x$, whose maximum is 1 (D).

Q7. (A, B)

Divide inside: the integrand equals $\dfrac{\left(1 - \frac{1}{x^2}\right)dx}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 1}}$ (for $x > 0$). With $u = x + \frac{1}{x}$ this is $\displaystyle\int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u|$, so $f(x) = x + \frac{1}{x}$. f has a local maximum -2 at $x = -1$ and a local minimum 2 at $x = 1$; its range is $(-\infty, -2] \cup [2, \infty)$ (C false); f increases for $x > 1$, so $f(\pi) > f(e)$ (D false).

Q8. (A, B, C)

$$\frac{d}{dx}\left(\frac{e^x}{\sqrt{1+x^2}}\right) = e^x\left(\frac{1}{\sqrt{1+x^2}} - \frac{x}{(1+x^2)^{3/2}}\right) = e^x\frac{x^2 - x + 1}{(1+x^2)^{3/2}}.$$

So $f(x) = \dfrac{1}{\sqrt{1+x^2}}$ (and $f(0) = 1$): even, bounded, range $(0, 1]$, with a single maximum at $x = 0$.

Q9. (A, B, C, D)

$\dfrac{d}{dx}(x^9 y) = 9x^8 y + x^9 y' = \dfrac{9x^8y^2 + x^9 yy'}{y}$. With $y^2 = 3x^2 + 2x + 1$ and $yy' = 3x + 1$:

$$\frac{d}{dx}\left(x^9 y\right) = \frac{9x^8(3x^2 + 2x + 1) + x^9(3x+1)}{y} = \frac{30x^{10} + 19x^9 + 9x^8}{y}.$$

So $a = 30$, $b = 19$, $c = 9$: $a + b = 49$, $b + c = 28$, $a - b + c = 20$, $b^2 - c^2 = 280$.

Q10. 5

Let $u = e^x - 1 - x$. The denominator is $1 + u^2 - \cos^2 x = u^2 + \sin^2 x$ and the numerator is $u' \sin x - u \cos x$. Hence the integrand is $\dfrac{d}{dx}\tan^{-1}\left(\dfrac{u}{\sin x}\right)$: $f = \tan^{-1}$ and $g(x) = \dfrac{e^x - 1 - x}{\sin x} \to 0$. Then $\dfrac{\tan^{-1} x}{x} \to 1$ and $\dfrac{g(x)}{x} = \dfrac{e^x - 1 - x}{x \sin x} \to \dfrac{1}{2}$, so $\dfrac{m}{n} = \dfrac{3}{2}$ and $m + n = \boxed{5}$.

Q11. 4

With $s = \sin x + \cos x$:

$$\sec x + \tan x + \operatorname{cosec} x + \cot x = \frac{1+\sin x}{\cos x} + \frac{1+\cos x}{\sin x} = \frac{1+s}{\sin x \cos x} = \frac{2(1+s)}{s^2-1} = \frac{2}{s-1}.$$

So the integrand is $\dfrac{(s-1)^2}{4} = \dfrac{2+\sin 2x - 2s}{4} = \dfrac{1}{2} + \dfrac{\sin 2x}{4} - \dfrac{\sin x + \cos x}{2}$, and

$$\int = \frac{x}{2} - \frac{\cos 2x}{8} + \frac{\cos x - \sin x}{2} = \frac{x}{2} + \frac{\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)}{2} + \frac{\cos 2x}{-8} + d.$$

Thus $a = 2$, $b = 2$, $c = -8$ and $|a+b+c| = \boxed{4}$.

Q12. 8

Put $y = \tan\theta$: the integral becomes

$$\int \frac{\cos^6\theta}{\sin^2\theta} d\theta = \int \left(\frac{1}{\sin^2\theta} - 3 + 3\sin^2\theta - \sin^4\theta\right) d\theta$$

$$= -\cot\theta - \frac{15}{8}\theta - \frac{1}{2}\sin 2\theta - \frac{1}{32}\sin 4\theta + C.$$

So $P = -1$, $Q = -\frac{15}{8}$, $R = -\frac{1}{2}$, $S = 2$, $T = -\frac{1}{32}$ and $P - \frac{8}{3}Q - 2R + S - 32T = -1 + 5 + 1 + 2 + 1 = \boxed{8}$.

Q13. 5

$x(x+3)(x+1)(x+2)+1 = (x^2+3x)(x^2+3x+2)+1 = (x^2+3x+1)^2$, and $2x+3$ is the derivative of x^2+3x+1. So the integral is $c - \dfrac{1}{x^2+3x+1}$: $f(x) = x^2+3x+1$ (with $f(0)=1$) and $f(1) = \boxed{5}$.

Q14. 0

Let $c = \cos x$, so $dc = -\sin x\, dx$ and $\tan x\sqrt{\sec x} = \dfrac{\sin x}{c^{3/2}}$. Since $\cos x + \sin^2 x = 1 + c - c^2$,

$$\int = -\int \frac{(1+c^2)\,dc}{c^{3/2}\sqrt{1+c-c^2}} = -\int \frac{\left(1+\frac{1}{c^2}\right)dc}{\sqrt{\frac{1}{c}-c+1}}.$$

With $w = \frac{1}{c} - c + 1$, $dw = -\left(\frac{1}{c^2}+1\right)dc$, so the integral is $2\sqrt{w}$, i.e. $2\sqrt{\sec x - \cos x + 1}$. Hence $f(x) = \sec x - \cos x + k$; $f\left(\frac{\pi}{3}\right) = 2 - \frac{1}{2} + k = \frac{3}{2}$ gives $k = 0$. Then $f\left(\frac{\pi}{4}\right) = \sqrt{2} - \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ and $\left[f\left(\frac{\pi}{4}\right)\right] = \boxed{0}$.

Q15. 2

The denominator is $\frac{\pi}{2}$, so the integrand is $\dfrac{4}{\pi}\sin^{-1}\sqrt{x} - 1$. With $s = \sin^{-1}\sqrt{x}$,

$$\int \sin^{-1}\sqrt{x}\, dx = xs - \frac{1}{2}\int \sqrt{\frac{x}{1-x}}\, dx = -\frac{1}{2}(1-2x)s + \frac{1}{2}\sqrt{x}\sqrt{1-x}.$$

Multiplying by $\frac{4}{\pi}$: $\frac{2}{\pi}\left[-s(1-2x) + \sqrt{x}\sqrt{1-x}\right] - x$, so $A = \boxed{2}$.

Passage 16

16.1: Differentiating the right side and multiplying by x^3-1 gives $f(x)=x^2-x-3$, so $f(5)=\boxed{17}$.

16.2: With $t=\sin x$, the integrand is $\dfrac{1-\frac{6}{t}}{t^2-t+3}=\dfrac{t-6}{t\,(t^2-t+3)}=-\dfrac{2}{t}+\dfrac{2t-1}{t^2-t+3}$. So $g=\ln\dfrac{\sin^2x-\sin x+3}{\sin^2x}$ and $\lim\limits_{t\to\pi/2}g(t)=\ln 3\approx\boxed{1.10}$.

Passage 17

17.1: Put $\sin^2\theta=x$, $2\sin\theta\cos\theta\,d\theta=dx$:

$$\int_0^{\pi/2}\cos^{2m}\theta\sin^{2n}\theta\,d\theta=\frac{1}{2}\int_0^1(1-x)^{m-\frac{1}{2}}x^{n-\frac{1}{2}}dx=\frac{1}{2}\beta\left(m+\frac{1}{2},\ n+\frac{1}{2}\right).$$

So $u=v=w=2$ and $u+v+w=\boxed{6}$.

17.2: Put $\dfrac{x}{1+x}=z$: $\displaystyle\int_0^\infty\frac{x^{m-1}}{(1+x)^{m+n}}dx=\int_0^1 z^{m-1}(1-z)^{n-1}dz=\beta(m,n)$. Similarly the other integral is $\beta(n,m)=\beta(m,n)$, so $k=\boxed{1}$.

Q18

(A) With $t^3=\tan x$, $dx=\dfrac{3t^2\,dt}{1+t^6}$, so the integral is $\displaystyle\int\frac{3t^3\,dt}{1+t^6}=\frac{3}{2}\int\frac{u\,du}{1+u^3}$ ($u=t^2$). Partial fractions $\dfrac{u}{1+u^3}=-\dfrac{1}{3(u+1)}+\dfrac{u+1}{3\,(u^2-u+1)}$ give $\frac{1}{4}\ln\dfrac{u^2-u+1}{(u+1)^2}+\dfrac{\sqrt3}{2}\tan^{-1}\dfrac{2u-1}{\sqrt3}$: $A=\frac14$, $B=\frac{\sqrt3}{2}$, (p, s).

(B) With $c=\cos x$: $\displaystyle\int\frac{-(2-c^2)\,dc}{2c^2-1}=\frac{c}{2}-\frac{3}{4}\int\frac{dc}{c^2-\frac12}$, giving $A=\frac12$, $B=\frac{3}{4\sqrt2}$: (r, t).

(C) $\dfrac{1}{(x^2+1)\,(x^2+4)}=\dfrac13\left(\dfrac{1}{x^2+1}-\dfrac{1}{x^2+4}\right)$: $A=\frac13$, $B=-\frac16$, (q).

(D) Divide numerator and denominator by x^2 and put $u=x+\frac1x$: the integral becomes $-\displaystyle\int\frac{du}{u\sqrt{u^2-2}}=\frac{1}{\sqrt2}\sin^{-1}\frac{\sqrt2}{u}+\text{const}=\frac{1}{\sqrt2}\sin^{-1}\frac{\sqrt2x}{x^2+1}$. So $A\sqrt2=\frac{1}{\sqrt2}$, $A=\frac12$: (r).

Q19

(A) The integrand is $\operatorname{cosec}^2x-\dfrac{1}{1+x^2}$, so the integral is $-\cot x+\cot^{-1}x$ (up to a constant): $A=1$, $B=-1$: (p, q).

(B) With $u=x+\sqrt{x^2+2}$, $x=\dfrac{u^2-2}{2u}$, $dx=\dfrac{u^2+2}{2u^2}du$: $\displaystyle\int\sqrt{u}\cdot\frac{u^2+2}{2u^2}du=\frac13u^{3/2}-\frac{2}{u^{1/2}}$: $A=1$, $B=2$: (p, r).

(C) Differentiating the right side reproduces $\dfrac{\sqrt{2-x-x^2}}{x}$ exactly when $A=1$, $B=2$: (p, r).

(D) With $u=\tan^2x$: $\displaystyle\int\frac{du}{1+u^2}=\tan^{-1}\left(\tan^2x\right)=-\cot^{-1}\left(\tan^2x\right)+\text{const}$: $B=-1$, (q).