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ASHUTOSH KUMAR

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ASSIGNMENT P8.1

Vector Algebra & Vector Geometry

Focus Topics: *Linear dependence, scalar and vector products, vector geometry, scalar triple product and vector triple product.*

Section A: Single Correct Choice

[Q1–Q4] Each question has four options (A), (B), (C) and (D). Exactly ONE option is correct.

Q 1. If the vector $\hat{i}-\hat{j}+\hat{k}$ bisects the angle between the vector $\vec{a}$ and the vector $4\hat{i}+3\hat{j}$, and the unit vector in the direction of $\vec{a}$ is $x\hat{i}+y\hat{j}+z\hat{k}$, then $x-15y+5z$ is equal to

(A) 5 **(B)** 9 **(C)** 7 **(D)** 11

Q 2. The vectors

$$\vec{a}=(\tan x)\hat{i}+2(\tan y)\hat{j}+\hat{k}, \qquad \vec{b}=(\tan x)\hat{i}-(\tan y)\hat{j}-\hat{k}$$

are non-zero and perpendicular. Define

$$\vec{p}=(\cos x)\hat{i}+(\sin x)\hat{j}+(\sin y)\hat{k}, \qquad \vec{q}=(\cos x)\hat{i}-(\sin x)\hat{j}+(\sin y)\hat{k}.$$

Which statement is always true under the given condition?

(A) $\vec{p}$ and $\vec{q}$ are linearly dependent **(B)** $\vec{p}\perp\vec{a}$
(C) $\vec{q}\perp\vec{a}$ **(D)** $\vec{p}\perp\vec{q}$

Q 3. Let $\vec{a},\vec{b},\vec{c}$ be non-zero, non-coplanar vectors and let

$$\vec{r}_1=p\vec{a}+q\vec{b}+\vec{c}, \qquad \vec{r}_2=\vec{a}+p\vec{b}+q\vec{c}.$$

If $\vec{r}_1+2\vec{r}_2$ and $2\vec{r}_1+\vec{r}_2$ are collinear, then (p,q) is

(A) $(0,0)$ **(B)** $(1,-1)$ **(C)** $(-1,1)$ **(D)** $(1,1)$

Q 4. Four vectors $\vec{a},\vec{b},\vec{c},\vec{x}$ satisfy

$$(\vec{a}\cdot\vec{x})\vec{b}=\vec{c}+\vec{x}, \qquad \vec{a}\cdot\vec{b}\neq 1.$$

Then $\vec{x}$ equals

(A) $\dfrac{(\vec{a}\cdot\vec{c})\vec{b}-\vec{c}(\vec{a}\cdot\vec{b}-1)}{\vec{a}\cdot\vec{b}-1}$ **(B)** $\dfrac{2(\vec{a}\cdot\vec{c})\vec{b}-\vec{c}}{\vec{a}\cdot\vec{b}-1}$

(C) $\dfrac{(\vec{a}\cdot\vec{c})\vec{b}+\vec{c}(\vec{a}\cdot\vec{b}-1)}{\vec{a}\cdot\vec{b}-1}$ **(D)** $\dfrac{2(\vec{a}\cdot\vec{c})\vec{c}+\vec{b}}{\vec{a}\cdot\vec{b}-1}$

Section B: One or More than One Correct

[Q5–Q9] Each question has four options (A), (B), (C) and (D). ONE OR MORE options may be correct.

Q 5. A unit vector coplanar with $\hat{i}+\hat{j}-\hat{k}$ and $\hat{i}-2\hat{j}+\hat{k}$, and perpendicular to $\hat{i}+\hat{j}+\hat{k}$, can be

(A) $\frac{1}{\sqrt{6}}(\hat{i}-2\hat{j}+\hat{k})$ **(B)** $\frac{1}{\sqrt{6}}(2\hat{i}-\hat{j}-\hat{k})$

(C) $\frac{1}{\sqrt{6}}(-\hat{i}+2\hat{j}-\hat{k})$ **(D)** $\frac{1}{\sqrt{6}}(-2\hat{i}+\hat{j}+\hat{k})$

Q 6. Let $U=\{\vec{a}_1,\vec{a}_2,\ldots,\vec{a}_n\}$ be a set of vectors. If $A\subseteq U$ is linearly dependent and $B\subseteq U$ is linearly independent, which statements are correct?

(A) Every subset of A is linearly dependent.

(B) Every superset of A is linearly dependent.

(C) Every subset of B is linearly independent.

(D) Every superset of B is linearly independent.

Q 7. Let

$$\vec{a}=9\hat{i}+\hat{j}-7\hat{k},\quad \vec{b}=2\hat{i}+3\hat{j}+3\hat{k},\quad \vec{c}=\hat{i}-2\hat{j}+\hat{k}.$$

Which statements are correct?

(A) $\vec{a},\vec{b},\vec{c}$ are linearly dependent.

(B) The orthogonal projection of $\vec{b}$ on $\vec{c}$ is $\dfrac{-\hat{i}+2\hat{j}-\hat{k}}{6}$.

(C) $\vec{a},\vec{b},\vec{c}$ are linearly independent.

(D) The orthogonal projection of $\vec{a}$ on the plane perpendicular to $\vec{b}$ is $\frac{5}{3}\hat{i}+\frac{2}{3}\hat{j}+\frac{1}{3}\hat{k}$.

Q 8. If $\vec{a}+2\vec{b}+3\vec{c}=\vec{0}$, then which of the following expressions are equal to $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}$?

(A) $2(\vec{a}\times\vec{b})$ **(B)** $6(\vec{b}\times\vec{c})$ **(C)** $3(\vec{c}\times\vec{a})$ **(D)** $\vec{0}$

Q 9. Consider the three planes

$$\vec{r}\cdot\vec{n}_1=a,\qquad \vec{r}\cdot\vec{n}_2=b,\qquad \vec{r}\cdot\vec{n}_3=c,\qquad abc\neq 0.$$

Define

$$\vec{S}=a(\vec{n}_2\times\vec{n}_3)+b(\vec{n}_3\times\vec{n}_1)+c(\vec{n}_1\times\vec{n}_2).$$

Which statements are correct?

(A) If the planes share a common line, then $\vec{S} = \vec{0}$.

(B) If the planes share a common line, then $\vec{S} \neq \vec{0}$.

(C) If $[\vec{n}_1, \vec{n}_2, \vec{n}_3] \neq 0$, the planes have a unique common point.

(D) If the normals are non-coplanar, the position vector of the common point is $\dfrac{\vec{S}}{[\vec{n}_1, \vec{n}_2, \vec{n}_3]}$.

Section C: Numerical Answer Type

[Q10–Q15] The answer to each question is a numerical value: an integer, or a decimal rounded off to two decimal places.

Q 10. Let $\vec{a}, \vec{b}$ be non-zero, non-collinear vectors such that $|\vec{a}| = 2$, $\vec{a} \cdot \vec{b} = 1$, and the angle between them is $\pi/3$. If $\vec{r}$ satisfies

$$\vec{r} \cdot \vec{a} = 2\vec{r} \cdot \vec{b} = 8(\vec{r} + 2\vec{a} - 10\vec{b}) \cdot (\vec{a} \times \vec{b}) = 4\sqrt{3}$$

and $\vec{r} + 2\vec{a} - 10\vec{b} = \lambda(\vec{a} \times \vec{b})$, find $\dfrac{1}{3\lambda^2}$._____

Q 11. A three-dimensional vector $\vec{V}$ satisfies

$$2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j}) = 2\hat{i} + \hat{k}.$$

If $3|\vec{V}| = \sqrt{m}$, find m._____

Q 12. Let $\vec{a}, \vec{b}, \vec{c}$ be non-zero, non-coplanar vectors and $\vec{d}$ another vector. If $[\vec{b}, \vec{c}, \vec{d}] = 24$ and

$$(\vec{a} \times \vec{b}) \times (\vec{c} \times \vec{d}) + (\vec{a} \times \vec{c}) \times (\vec{d} \times \vec{b}) + (\vec{a} \times \vec{d}) \times (\vec{b} \times \vec{c}) + k\vec{a} = \vec{0},$$

find $k/8$._____

Q 13. Let $\vec{a}, \vec{b}, \vec{c}$ be vectors satisfying $[\vec{a}, \vec{b}, \vec{c}] = 1$, $\vec{c} = \lambda(\vec{a} \times \vec{b})$, $|\vec{a}| = \sqrt{2}$, $|\vec{b}| = \sqrt{3}$, and $|\vec{c}| = 1/\sqrt{3}$. If the angle between $\vec{a}$ and $\vec{b}$ is π/k, where k is a positive integer, find k._____

Q 14. Let $a_1, a_2, \ldots, a_n$ $(n \geq 10)$ be natural numbers in arithmetic progression. Define

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k},$$
$$\vec{b} = a_2\hat{i} + a_3\hat{j} + a_1\hat{k},$$
$$\vec{c} = a_3\hat{i} + a_1\hat{j} + a_2\hat{k}.$$

If the volume of the parallelepiped formed by $\vec{a}, \vec{b}, \vec{c}$ is 54, find a_6._____

Q 15. Let

$$\vec{f}(x) = (\cos x)\hat{i} + (\sin x)\hat{j} + (\cos 2x)\hat{k},$$
$$\vec{g}(x) = (\tan x)\hat{i} + (\sin 3x)\hat{j} + (\cos 4x)\hat{k},$$
$$\vec{h}(x) = (\cos 3x)\hat{i} + (\sin 5x)\hat{j} + (\cos 6x)\hat{k}.$$

Evaluate $\int_0^{\pi} [\vec{f}(x), \vec{g}(x), \vec{h}(x)]\, dx$, where $[\cdot, \cdot, \cdot]$ denotes the scalar triple product._____

Section D: Linked Comprehension Type

[Q16–Q17] Answer both questions using the same given passage.

Passage. Let $\vec{a}$ and $\vec{b}$ be position vectors of points A and B respectively, and let $\vec{c}$ be the position vector of a variable point in the plane of triangle OAB, where O is the origin.

Q 16. If $\vec{a} = 3\hat{i} - 3\hat{k}$, $\vec{b} = 2\hat{i} + 2\hat{j} + 2\hat{k}$, and $\vec{c} = \hat{j} + 2\hat{k}$, then C lies

(A) inside $\triangle OAB$

(B) outside $\triangle OAB$ but inside $\angle AOB$

(C) outside $\triangle OAB$ but inside $\angle OAB$

(D) outside $\triangle OAB$ but inside $\angle OBA$

Q 17. If $\vec{a} = \hat{i} + \hat{j}$, $\vec{b} = \hat{j} + \hat{k}$, and $\vec{c} = \hat{i} + 3\hat{j} + 2\hat{k}$, then C lies

(A) inside $\triangle OAB$

(B) outside $\triangle OAB$ but inside $\angle AOB$

(C) outside $\triangle OAB$ but inside $\angle OAB$

(D) outside $\triangle OAB$ but inside $\angle OBA$

Section E: Matrix Match Type

[Q18–Q19] Match each item in List-I with the corresponding item in List-II. Select the code containing all four matches.

Q 18. Match the plane descriptions in List-I with their vector equations in List-II.

List-I	List-II
(A) Plane perpendicular to the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z+1}{2}$ and passing through $(3,6,2)$	(P) $\vec{r}\cdot(\hat{i}-3\hat{j}-2\hat{k})=3$
(B) Plane through $(5,-2,4)$ and parallel to $4x-12y-8z=7$	(Q) $\vec{r}\cdot(\hat{i}-\hat{j}-\hat{k})=2$
(C) Plane containing $\vec{r}=2\hat{i}+\lambda(\hat{j}-\hat{k})$ and perpendicular to $\vec{r}\cdot(\hat{i}+\hat{k})=3$	(R) $\vec{r}\cdot(\hat{i}-\hat{j}-\hat{k})=0$
(D) Plane containing the lines $\frac{x-1}{1}=\frac{y-1}{2}=\frac{z}{-1}$ and $\frac{x-1}{-1}=\frac{y-1}{1}=\frac{z}{-2}$	(S) $\vec{r}\cdot(\hat{i}-\hat{j}+2\hat{k})=1$

(A) $A\to P,\ B\to Q,\ C\to R,\ D\to S$

(B) $A\to S,\ B\to P,\ C\to Q,\ D\to R$

(C) $A\to R,\ B\to P,\ C\to S,\ D\to Q$

(D) $A\to S,\ B\to Q,\ C\to P,\ D\to R$

Q 19. Match each pair of planes in List-I with its line of intersection in List-II.

List-I	List-II
(A) $x+y=0,\quad y+z=0$	(P) $x=2,\quad y=2007-t,\quad z=t-2004$
(B) $x-2=0,\quad y-3=0$	(Q) $x=2,\quad y=-t,\quad z=t+1$
(C) $x-2=0,\quad y+z-3=0$	(R) $x=t,\quad y=-t,\quad z=t$
(D) $x-2=0,\quad x+y+z-3=0$	(S) $x=2,\quad y=3,\quad z=t$

(A) $A\to P,\ B\to R,\ C\to Q,\ D\to S$

(B) $A\to R,\ B\to S,\ C\to P,\ D\to Q$

(C) $A\to S,\ B\to Q,\ C\to P,\ D\to R$

(D) $A\to R,\ B\to P,\ C\to S,\ D\to Q$

■

Answer Key: Assignment P8.1

ANSWER KEY

1. D
2. D
3. D
4. A
5. AC
6. BC
7. BC
8. ABC
9. ACD
10. 4
11. 6
12. 6
13. 4
14. 10
15. 0
16. C
17. B
18. (A) → (s) (B) → (p) (C) → (q) (D) → (r)
19. (A) → (r) (B) → (s) (C) → (p) (D) → (q)

Hints and Solutions: Assignment P8.1

Q1.

The unit vector along the bisector must be chosen consistently with the given angle-bisector direction. The bisector condition gives $(x, y, z) = \left(-\frac{2}{3}, -\frac{11}{15}, \frac{2}{15}\right)$. Hence $x - 15y + 5z = -\frac{2}{3} + 11 + \frac{2}{3} = 11$.

Q2.

Perpendicularity gives $\tan^2 x - 2\tan^2 y - 1 = 0$. Rearranging and using the tangent-to-double-angle identity yields $\cos 2x = -\sin^2 y$. Therefore $\vec{p} \cdot \vec{q} = \cos^2 x - \sin^2 x + \sin^2 y = \cos 2x + \sin^2 y = 0$, so option D is correct.

Q3.

Since $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar, compare their coefficients. For $(p, q) = (1, 1)$, $\vec{r}_1 = \vec{r}_2 = \vec{a} + \vec{b} + \vec{c}$, so the two specified vectors coincide. The other offered pairs fail coefficient proportionality. Thus D.

Q4.

Dot $(\vec{a}\cdot\vec{x})\vec{b} = \vec{c} + \vec{x}$ with $\vec{a}$ to get $(\vec{a}\cdot\vec{x})(\vec{a}\cdot\vec{b} - 1) = \vec{a}\cdot\vec{c}$. Substituting $(\vec{a}\cdot\vec{x}) = \dfrac{\vec{a}\cdot\vec{c}}{\vec{a}\cdot\vec{b} - 1}$ back into the original equation gives option A.

Q5.

The vector $\hat{i} - 2\hat{j} + \hat{k}$ is perpendicular to $\hat{i} + \hat{j} + \hat{k}$ and lies in the plane spanned by the two given vectors. Its magnitude is $\sqrt{6}$. Both it and its negative are unit vectors satisfying the conditions. Hence A and C.

Q6.

A superset of a linearly dependent set remains dependent, so B is true. Every subset of a linearly independent set is independent, so C is true. A and D are false in general.

Q7.

Since $\vec{b} \cdot \vec{c} = -1$ and $|\vec{c}|^2 = 6$, the projection of $\vec{b}$ on $\vec{c}$ is $\dfrac{\vec{b}\cdot\vec{c}}{|\vec{c}|^2}\vec{c} = -\vec{c}/6$, giving B. Also $\vec{a} = \vec{b} \times \vec{c}$, so $\vec{a} \cdot (\vec{b} \times \vec{c}) = |\vec{a}|^2 \neq 0$; the vectors are linearly independent, giving C. A and D are false.

Q8.

From $\vec{a} + 2\vec{b} + 3\vec{c} = \vec{0}$, we get $\vec{b} \times \vec{c} = \frac{1}{3}(\vec{a} \times \vec{b})$ and $\vec{c} \times \vec{a} = \frac{2}{3}(\vec{a} \times \vec{b})$. Thus the required sum is $2(\vec{a} \times \vec{b}) = 6(\vec{b} \times \vec{c}) = 3(\vec{c} \times \vec{a})$. Hence A, B and C.

Q9.

For three planes sharing a common line, the compatibility condition forces $a(\vec{n}_2 \times \vec{n}_3) + b(\vec{n}_3 \times \vec{n}_1) + c(\vec{n}_1 \times \vec{n}_2) = \vec{0}$, so A. A non-zero scalar triple product means the normals are linearly independent, giving a unique intersection point (C); the displayed vector formula in D is Cramer's rule for that point. Hence A, C and D.

Q10.

The angle and dot product give $|\vec{b}| = 1$ and $|\vec{a} \times \vec{b}|^2 = 3$. From $\vec{r} + 2\vec{a} - 10\vec{b} = \lambda(\vec{a} \times \vec{b})$ and the equality chain, $8\lambda|\vec{a} \times \vec{b}|^2 = 4\sqrt{3}$, so $\lambda = \sqrt{3}/6$. Therefore $1/(3\lambda^2) = 4$.

Q11.

Write $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k}$. Comparing coordinates in $2\vec{V} + \vec{V} \times (\hat{i} + 2\hat{j}) = 2\hat{i} + \hat{k}$ gives $(x, y, z) = (7/9, 1/9, -2/9)$. Hence $|\vec{V}|^2 = 2/3$, so $(3|\vec{V}|)^2 = 6$ and $m = 6$.

Q12.

Using the vector triple-product identity on the first three terms gives $-2[\vec{b}, \vec{c}, \vec{d}]\vec{a} = -48\vec{a}$. The equation therefore requires $k = 48$, so $k/8 = 6$.

Q13.

Since $\vec{c} = \lambda(\vec{a} \times \vec{b})$ and $[\vec{a}, \vec{b}, \vec{c}] = 1$, we obtain $\lambda|\vec{a} \times \vec{b}|^2 = 1$. Together with $|\vec{c}|^2 = 1/3$ this gives $\sin^2\theta = 1/2$, where θ is the angle between $\vec{a}$ and $\vec{b}$. Taking θ acute, $\theta = \pi/4$ and $k = 4$.

Q14.

If $a_1 = a$ and the common difference is d, the determinant magnitude for the cyclic triple is $9(a + d)d^2 = 54$. Hence $(a + d)d^2 = 6$. Since a, d are positive integers, $d = 1$ and $a = 5$, giving $a_6 = 10$.

Q15.

Let $D(x) = [\vec{f}(x), \vec{g}(x), \vec{h}(x)]$. The determinant has the antisymmetry $D(\pi - x) = -D(x)$; pairing the integrand at x and $\pi - x$ makes the integral over $[0, \pi]$ equal to zero. The apparent singularity in $\tan x$ at $x = \pi/2$ is removable in the scalar-triple-product expression.

Q16.

With $\vec{a} = (3, 0, -3)$, $\vec{b} = (2, 2, 2)$ and $\vec{c} = (0, 1, 2)$, coefficient comparison gives $\vec{c} = -\frac{1}{3}\vec{a} + \frac{1}{2}\vec{b}$. The coefficients sum to $1/6 < 1$, but the coefficient of $\vec{a}$ is negative; hence C is outside the triangle but inside $\angle OAB$. Option C.

Q17.

Here $\vec{c} = \vec{a} + 2\vec{b}$. Both coefficients are positive, placing C inside the angle AOB, while their sum is $3 > 1$, placing it outside triangle OAB. Option B.

Q18.

Match each plane separately. (A) has normal $(1, -1, 2)$ and passes through $(3, 6, 2)$, so $\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 1$; thus $A \to S$. (B) has normal proportional to $(1, -3, -2)$ and its constant is 3, so $B \to P$. (C) has normal $(1, -1, -1)$ and passes through $(2, 0, 0)$, giving $\vec{r} \cdot (\hat{i} - \hat{j} - \hat{k}) = 2$; thus $C \to Q$. (D) gives the plane $x - y - z = 0$, so $D \to R$.

Q19.

Determine the line from the two plane equations in each case. (A) gives $x = -y = z$, so $A \to R$. (B) fixes $x = 2$ and $y = 3$ while z is free, so $B \to S$. (C) gives $x = 2$ and $y + z = 3$, which is line P; therefore $C \to P$. (D) gives $x = 2$ and $y + z = 1$, which is line Q; therefore $D \to Q$.